Lecture 2: summary

$$\underset{\text{Abelian}}{\mathcal{E}} \subset \underset{\text{dg}}{\mathcal{D}}$$

$$\mathfrak{M}_{\mathcal{E}} \underset{\text{open}}{\subset} \mathfrak{M}_{\mathcal{D}}$$

$\mathcal{E} = \mathrm{rep}\, \Pi_Q$ preprojective algebra

$\mathcal{E} = \mathrm{Coh}^{H\text{-}ss}_{p(t)}(S)$ S: K3 or Abelian

$\mathcal{E} = \mathrm{rep}\, \pi_1$ (Riemann surface)

$\mathcal{E} = \mathrm{rep}\, \Lambda_Q$ multiplicative preprojective algebra

2CY condition: to have local description by preprojective algebras and some compatibility of RHom complexes

$$JH_{\mathcal{E}} : \mathfrak{M}_{\mathcal{E}} \to \mathcal{M}_{\mathcal{E}}$$

$$\mathcal{A}_{\mathcal{E}} := JH_* \mathbb{D}\mathbb{Q}^{vir}_{\mathfrak{M}_{\mathcal{E}}} \in \mathcal{D}^+_c(\mathcal{M}_{\mathcal{E}})$$ sheafified CoHA

$$^{p}\mathcal{H}^0(\mathcal{A}_{\mathcal{E}}) =: \mathcal{BPS}_{\mathcal{E},\mathrm{Alg}} \in \mathrm{Perv}(\mathcal{M}_{\mathcal{E}})$$ sheafified BPS algebra

CoHAs for quivers with potential

$\mathcal{A}_{\Pi_Q}$ recovered from $\mathcal{A}_{\tilde{Q},W} \in \mathcal{D}^+_c(\mathcal{M}_{\tilde{Q}})$

$\in \mathcal{D}^+_c(\mathcal{M}_{\Pi_Q}) \hookrightarrow \mathcal{D}^+_c(\mathcal{M}_{\overline{Q}})$ via $\mathcal{M}_{\tilde{Q}} \to \mathcal{M}_{\overline{Q}}$.

$(\tilde{Q}, W) \rightsquigarrow$ Lie algebra $\mathcal{BPS}_{\tilde{Q},W} \in \mathrm{Perv}(\mathcal{M}_{\tilde{Q}})$

$\rightsquigarrow$ Lie algebra $\mathcal{BPS}^{3d}_{\Pi_Q,\mathrm{Lie}} \in \mathrm{Perv}(\mathcal{M}_{\Pi_Q})$.

PBW theorem: $$\mathrm{Sym}_{\boxdot}\left(\mathcal{BPS}^{3d}_{\Pi_Q,\mathrm{Lie}} \otimes H^*_{\mathbb{C}^*}(pt)\right) \xrightarrow{\sim} \mathcal{A}_{\Pi_Q}$$

$\mathcal{A}_{\Pi_Q}$ is a semisimple complex

$\mathcal{BPS}^{3d}_{\Pi_Q, Lie} \hookrightarrow \mathcal{BPS}_{\Pi_Q, Alg}$ are semisimple perverse sheaves.

$$\mathcal{U}\left(\mathcal{BPS}^{3d}_{\Pi_Q, Lie}\right) \xrightarrow{\sim} \mathcal{BPS}_{\Pi_Q, Alg}.$$

<u>Lecture 3</u> : V - Generalized Kac-Moody algebras

VI - Generators and relations for the BPS algebra and PBW isomorphism.

Motivation of this lecture

(Q,W) quiver with potential
Q symmetric (incidence matrix is symmetric)

$\rightsquigarrow$ $BPS_{Q,W}$ is a $\mathbb{N}^{Q_0} \times \mathbb{Z}$-graded Lie algebra.

What is it? Can we describe it by generators and relations?

$\to$ Not obvious, very few examples.

① Q symmetric, $W=0$: $BPS_{Q,W}$ is an Abelian Lie algebra

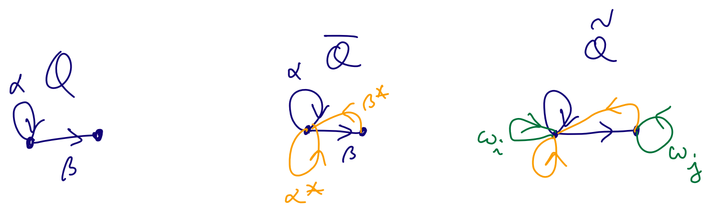
② $(\tilde{Q},W)$ tripled quiver with canonical potential: if Q is finite type,

$BPS_{\tilde{Q},W} \cong \mathfrak{n}_Q^+$ positive part of KM-algebra associated to Q

if Q is affine, $BPS_{\tilde{Q},W} \cong \mathfrak{n}_Q^+[u]$ + linear extension of the Lie bracket

Saying what $BPS_{\tilde{Q},W}$ is is very hard!

<u>Today</u>: Generalize this for any tripled quiver with potential.
[more generally, any 3CY completion of a 2CY category]



$\rho = [\alpha,\alpha^*] + [\beta,\beta^*]$

$W = (\omega_i + \omega_j)\cdot \rho$ canonical cubic potential.

V - Generalised Kac-Moody algebras

① GKMs

$M = \mathbb{N}^{Q_0}$ monoid (for simplicity) $M \subset \mathbb{N}^{Q_0}$ works the same way.

$M \times M \xrightarrow{(-,-)} \mathbb{Z}$ symmetric, bilinear

$\Phi^+ \subset M \setminus \{0\}$ subset of simple positive roots

<u>Assumptions</u> : $\forall d, d' \in \Phi^+, \ (d, d') \leq 0$

$\forall d \in \Phi^+, \ (d,d) > 0 \Rightarrow (d,d) = 2$

$\mathcal{G}$ $\Phi^+ \times \mathbb{Z}$-graded vector space, finite dimensional graded pieces.
"positive Chevalley generators".

$\mathcal{G}^\vee$ graded dual vector space

$\mathfrak{g}$ is the Lie algebra generated by $\mathcal{G} \oplus \underset{\underset{\mathfrak{h}}{\parallel}}{\mathbb{Q}^{Q_0}} \oplus \mathcal{G}^\vee$ with the relations

$$[h, h'] = 0 \qquad \forall h, h' \in \mathfrak{h}$$

$$[h, \alpha_i^\vee] = \pm (h, h_i) \alpha_i^\vee \qquad \alpha_i^\vee \in \mathcal{G}_i^\vee$$

$$[\alpha_i, \alpha_j^\vee] = \delta_{ij} \ \alpha_j^\vee(\alpha_i) \ h_i$$

$$[\alpha_i^\vee, -]^{1-(h_i, h_j)} (\alpha_j^\vee) = 0 \qquad \text{if } (h_i, h_j) = 0 \text{ or } (h_i, h_i) = 2$$

(Serre relations)

Properties Triangular decomposition

$$\mathfrak{g} \simeq \underset{\langle e_i^\vee \rangle}{\mathfrak{n}^-} \oplus \mathfrak{h} \oplus \underset{\langle e_i \rangle}{\mathfrak{n}^+}.$$

$i \in \Phi^+ \subset \mathbb{N}^{Q_0}$
$\rightarrow h_i \in \mathfrak{h} = \mathbb{Q}^{Q_0}$

Trichotomy of roots :

h_i, $i \in \Phi^+$ come in 3 kinds

- real roots : $(h_i, h_i) = 2$
- imaginary roots : $(h_i, h_i) = 0$
- hyperbolic roots : $(h_i, h_i) < 0$.

Positive part : $\mathfrak{n}^+$ is generated by e_i with the Serre relations.

Examples of GKMs

$Q = (Q_0, Q_1)$ quiver

$\mathfrak{h} = \mathbb{C}^{Q_0}$

Q_0 vertices $\qquad$ $s, t : Q_1 \longrightarrow Q_0$ source and target maps

Q_1 arrows

$$(d, e) = \chi_Q(d, e) + \chi_Q(e, d) \quad \text{symmetrized Euler form}$$

$$= 2 \sum_{i \in Q_0} d_i e_i - \sum_{\alpha \in Q_1} \left(d_{s(\alpha)} e_{t(\alpha)} + e_{s(\alpha)} d_{t(\alpha)} \right)$$

$\boxed{1}$ Q has no loops

$\Phi^+ = Q_0 \subset \mathbb{N}^{Q_0}$

$\mathfrak{g}_i = \mathbb{Q}$ $\quad i \in Q_0$

$\mathfrak{g} = \mathfrak{g}_Q$ is the Kac-Moody algebra classically associated to Q.

$\boxed{3/2}$ Q has possible loops

$\Phi^+ = Q_0^{real} \sqcup (Q_0^{im} \times \mathbb{Z}_{\geq 1}) \subset \mathbb{N}^{Q_0}$

$\mathfrak{g}_i = \mathbb{Q}$

$\rightsquigarrow$ the GKM associated to Q by Bozec in his PhD thesis

$\boxed{2}$ $Q =$  $\quad \Phi^+ = \mathbb{Z}_{\geq 1} \subset \mathbb{Q} =: \mathfrak{h}$

$\mathfrak{g}_n = \mathbb{Q}[2] \quad \forall n \geq 1$

$\Rightarrow \mathfrak{g} = \mathfrak{heis}$, the Heisenberg Lie algebra acting on the coh. of points on $\mathbb{C}^2$.

These algebras appear naturally from geometry and act on many highly important moduli spaces

② Positive part of GKMs in categories of perverse sheaves

$\mathcal{M}$ monoid in schemes s.t. $\pi_0(\mathcal{M}) = M \subset \mathbb{N}^{Q_0}$ (Q_0 finite)

$\mathcal{G} = \bigoplus_{d \in \Phi^+} \mathcal{G}_d \in \mathrm{Perv}(\mathcal{M})$ "generating perverse sheaf"

$\oplus : \mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{M}$ finite morphism.

More assumptions: $\mathcal{M}_d = \mathrm{pt}$ and $\mathcal{G}_d = \mathbb{Q}_{\mathrm{pt}}$ if $(d,d) = 2$.
(real roots)

$\mathfrak{n}^+$ Lie algebra object in $\mathrm{Perv}(\mathcal{M})$ generated by $\mathcal{G}$ modulo the Serre ideal, generated by the Serre relations

$$\begin{cases} \mathrm{ad}(\mathcal{G}_d)(\mathcal{G}_{d'}) & \text{if } (d,d') = 0 \\ \mathrm{ad}(\mathcal{G}_d)^{1-(d,d')}(\mathcal{G}_{d'}) & \text{if } (d,d) = 2 \end{cases}$$

These Serre relations generate the Serre ideals

$$\mathcal{I}_{\mathcal{A},\mathrm{Lie}} \subset \mathrm{Free}_{\boxdot,\mathrm{Lie}}(\mathcal{G}) \in \mathrm{Perv}(\mathcal{M})$$

$$\mathcal{I}_{\mathcal{A},\mathrm{Alg}} \subset \mathrm{Free}_{\boxdot}(\mathcal{G}) = \text{free tensor algebra generated by } \mathcal{G}.$$

$\in \mathrm{Perv}(\mathcal{M})$

VI- The BPS algebra by generators and relations and the PBW theorem

① Roots

$$\Sigma_{\mathcal{E}} = \{ a \in \pi_0(\mathcal{M}_{\mathcal{E}}) \mid \mathcal{M}_{\mathcal{E}} \text{ has a nonempty locus of simples} \}$$

$\Updownarrow$

$JH_{\mathcal{E},a} : \mathfrak{M}_{\mathcal{E},a} \to \mathcal{M}_{\mathcal{E},a}$ is a $\mathbb{G}_m$-gerbe

$$\Phi^+_{\mathcal{E}} = \Sigma_{\mathcal{E}} \cup \left\{ \ell a \; ; \; \begin{array}{l} a \in \Sigma_{\mathcal{E}} , \; (a,a)_{\mathcal{E}} = 0 \\ \ell \geq 2 \end{array} \right\}$$

$\pi_0(\mathcal{M}_{\mathcal{E}}) \times \pi_0(\mathcal{M}_{\mathcal{E}}) \longrightarrow \mathbb{Z}$

$(-,-)_{\mathcal{D}}$ Euler form of $\mathcal{D}$

$$(M,N)_{\mathcal{D}} = \sum_{i \in \mathbb{Z}} (-1)^i \operatorname{Hom}_{H^0(\mathcal{D})}(M, N[i])$$

② Generators

For each $a \in \Phi^+_{\mathcal{E}}$, we have a generating perverse sheaf $\mathcal{G}_a \in \operatorname{Perv}(\mathcal{M}_{\mathcal{E},a})$.

* $\mathcal{G}_a := \mathcal{IC}(\mathcal{M}_{\mathcal{E},a})$ if $a \in \Sigma_{\mathcal{E}}$

Define
$$\Delta_\ell : \mathcal{M}_{\mathcal{E},a} \longrightarrow \mathcal{M}_{\mathcal{E},\ell a}$$
$$x \longmapsto x^{\oplus \ell}$$

* $\mathcal{G}_a = (\Delta_\ell)_* \mathcal{IC}(\mathcal{M}_{\mathcal{E},a})$ if $\begin{array}{l} a \in \Sigma_{\mathcal{E}} \\ (a,a) = 0 \\ \ell \geq 2 \end{array}$.

Proposition: We have monomorphisms

$\mathcal{Y}_a \hookrightarrow BPS_{\mathcal{E},\mathrm{Alg}} \qquad a \in \Phi^+_{\mathcal{E}}.$

Proof: • If $a \in \Sigma_{\mathcal{E}}$, $JH : \mathfrak{M}_{\mathcal{E},a} \longrightarrow \mathcal{M}_{\mathcal{E},a}$ is a $\mathbb{G}_m$-gerbe

+ simple locus $\mathcal{M}^s_{\mathcal{E},a} \subset \mathcal{M}_{\mathcal{E},a}$ is smooth

$$\Rightarrow \quad \mathcal{A}_{\mathcal{E}}\big|_{\mathcal{M}^s_{\mathcal{E}}} \cong \mathbb{Q}_{\mathcal{M}^s_{\mathcal{E},a}}[\dim \mathcal{M}^s_{\mathcal{E},a}] \otimes H^*_{\mathbb{C}^*}(pt).$$

Conclude by semisimplicity of $\mathcal{A}_{\mathcal{E}}$.

• If $a \in \Sigma_{\mathcal{E}}$, $(a,a)=0$, then the monomorphism

(*) $\Delta_{\ell *} \mathcal{IC}(\mathcal{M}_{\mathcal{E},a}) \longrightarrow BPS_{\mathcal{E},\ell a}$ is trickier to exhibit.

$\mathcal{M}^{ss}_{\mathcal{E},\bullet a} \subset \mathcal{M}_{\mathcal{E}}$ open — locus of semisimple objects in $\mathcal{C}$ simple summands are of class $\mathbb{N}a \subset \pi_0(\mathcal{M}_{\mathcal{E}})$.

Fact: $$\mathcal{A}\big|_{\mathcal{M}^{ss}_{\mathcal{E},\bullet a}} \cong \mathrm{Sym}\Big(\bigoplus_{\ell\geq 1} (\Delta_\ell)_* \mathcal{IC}(\mathcal{M}_{\mathcal{E},a}) \otimes H^*_{\mathbb{C}^*}(pt)\Big)$$

This gives (*).

② The BPS algebra by gens & rels : Theorem A

Theorem A (Davison–H–Schlegel Mejia, 2023)

$$\mathcal{BPS}_{\mathcal{E},\mathrm{Alg}} \cong U(\mathfrak{n}^+_{\mathcal{E}}) \in \mathrm{Perv}(\mathcal{M}_{\mathcal{E}})$$

where $\mathfrak{n}^+_{\mathcal{E}} \in \mathrm{Perv}(\mathcal{M}_{\mathcal{E}})$ is the positive part of a GKM generated by

$$\begin{cases} \mathcal{IC}(\mathcal{M}_{\mathcal{E},a}) & a \in \Sigma_{\mathcal{E}} \\ \mathcal{IC}(\mathcal{M}_{\mathcal{E},a}) & a \in \Sigma_{\mathcal{E}},\ \ell \geq 2, \\ & (a,a)_{\mathcal{E}} = 0. \end{cases}$$

$$(-,-)_{\mathcal{E}} : \pi_0(\mathcal{M}_{\mathcal{E}}) \times \pi_0(\mathcal{M}_{\mathcal{E}}) \longrightarrow \mathbb{Z} \quad \text{Euler form}$$

determines the relations.

Corollary : $\mathcal{BPS}^{3d}_{\Pi_Q,\mathrm{Lie}} \in \mathrm{Perv}(\mathcal{M}_{\Pi_Q})$ is the positive part of a GKM

$\mathrm{BPS}_{\tilde{Q},W}$ is the positive part of a GKM.

3 The PBW theorem: Theorem B

Theorem B (Davison-H-Schlegel Mejia)

The PBW map

$$\mathrm{Sym}_{\boxdot}\left(\mathcal{R}^+_{\mathcal{E}} \otimes H^*_{\mathbb{C}^*}(pt)\right) \longrightarrow \mathcal{A}_{\mathcal{E}}$$

is an isomorphism in $\mathcal{D}^+_c(\mathcal{M}_{\mathcal{E}})$.

For $\mathcal{E} = \Pi_Q$, comes from dimensional reduction and PBW theorem for quivers with potential.

For any $\mathcal{E}$: local neighbourhood theorem + compatibility of multiplications.

How to proof these theorems

Theorem A:

We have canonical maps

$$\mathcal{Y}_a \longrightarrow \mathcal{BPS}_{\mathcal{E}} \qquad \forall a \in \Phi^+_{\mathcal{E}}.$$

Extend it to

$$\mathrm{Free}_{\boxdot\text{-Alg}}\Big(\underbrace{\bigoplus_{a\in\Phi^+_{\mathcal{E}}} \mathcal{Y}_a}_{\mathcal{Y} :=}\Big) \xrightarrow{\Phi} \mathcal{BPS}_{\mathcal{E}}.$$

$$\cup$$

$$\mathcal{J}_{\mathcal{E},\mathrm{Alg}} \quad \text{Serre ideal.}$$

1. Prove that Φ vanishes on the Serre ideal

(2) Prove that the induced map

$$\mathrm{Free}_{\boxdot\text{-Alg}}(\mathcal{G}) \big/ \mathcal{I}_{\mathcal{E},\mathrm{Alg}} \xrightarrow{\overline{\Phi}_{\mathcal{E}}} \mathcal{BPS}_{\mathcal{E},\mathrm{Alg}}$$

is an isomorphism.

(3) Notice this is a morphism between semisimple perverse sheaves.

(4) Reduce to preprojective algebras of quivers

(5) Prove the result for preprojective algebras of quivers.

Start of an induction proof.

If ϕ_{Π_Q} is not an isomorphism, then $\underbrace{\ker \phi_{\Pi_Q} \oplus \operatorname{coker} \phi_{\Pi_Q}}_{\mathcal{K}}$ is a semisimple perverse sheaf, $\neq 0$

$\rightsquigarrow$ Pick $x \in \mathcal{M}_{\Pi_Q,d}$ s.t $i_x^! \mathcal{K} \neq 0$

$$x \rightsquigarrow \bigoplus S_i^{m_i} \rightsquigarrow \underline{S} = \{S_1, \dots, S_r\}.$$

$\phi_{\Pi_{Q_{\underline{S}}}}$ is not an iso, ... (induction)

Key: Q quiver Π_Q preprojective algebra

$x \in \mathcal{M}_{\Pi_Q} \rightsquigarrow M = \bigoplus_{i=1}^{r} S_i^{m_i}$ semisimple rep of Π_Q, $d = \dim M$.

$\underline{S} := \{S_1, \ldots, S_r\}$ collection of simple objects of $\operatorname{rep} \Pi_Q$

$\overline{Q}_{\underline{S}}$ Ext quiver, (m_i) dimension vector.

Then, $(\overline{Q}_{\underline{S}}, (m_i)) \leq (Q, d)$ for some total order on $\{(Q, d) \mid Q \text{ quiver}, d \in \mathbb{N}^{Q_0} \text{ dimension vector}, \operatorname{supp}(d) = Q\}$.

Partial order: $(Q, d) \leq (Q', d')$

$$\Leftrightarrow \begin{cases} |d| \leq |d'| \text{ or} \\ |d| = |d'| \text{ and } \# Q_0 \geq Q_0' \end{cases}.$$

The terminal case of the induction is dealt with using the strictly seminilpotent CoHA: next time: this is a much simpler object

Theorem B: ① Construct the map

$$\mathrm{Sym}_{\boxdot}\left(\mathcal{IC}_{\mathcal{E}}^{+} \otimes H^{*}_{\mathbb{C}^{*}}\right) \xrightarrow{\varphi_{\mathcal{E}}} \mathcal{A}_{\mathcal{E}} \quad \text{in } \mathcal{D}_c^{+}(\mathcal{M}_{\mathcal{E}}).$$

<u>How</u>?

$$\begin{tikzcd} \mathcal{IC}_{\mathcal{E}}^{+} \arrow[r, hook] & \mathcal{BPS}_{\mathcal{E},\mathrm{Alg}} \arrow[rr, "\text{adjunction map}"'] & & \mathcal{A}_{\mathcal{E}} \arrow[loop below] \\ & & & H^{*}_{\mathbb{C}^{*}}(pt) \end{tikzcd}$$

action of the first Chern class of the determinant line bundle.

$$\rightsquigarrow \quad \mathcal{IC}_{\mathcal{E}}^{+} \otimes H^{*}_{\mathbb{C}^{*}}(pt) \longrightarrow \mathcal{A}_{\mathcal{E}}$$

Then

$$\begin{tikzcd} \mathrm{Sym}\left(\mathcal{IC}_{\mathcal{E}}^{+} \otimes H^{*}_{\mathbb{C}^{*}}(pt)\right) \arrow[rr, "\varphi_{\mathcal{E}}"] \arrow[d, hook] \arrow[drr, phantom, "\circlearrowright"] & & \mathcal{A}_{\mathcal{E}} \\ \mathrm{Free}_{\boxdot\text{-Alg}}\left(\mathcal{IC}_{\mathcal{E}}^{+} \otimes H^{*}_{\mathbb{C}^{*}}(pt)\right) \arrow[urr, "\text{use CoHA product}"'] & & \end{tikzcd}$$

② Show that φ is an isomorphism.
Suffices to show that:

$$\forall x \in \mathcal{M}_{\mathcal{E}}, \quad \{x\} \xhookrightarrow[i_x]{} \mathcal{M}_{\mathcal{E}},$$

$$i_x^{!}\, \varphi_{\mathcal{E}} \text{ is an isomorphism.}$$

Any such x corresponds to a semisimple object of $\mathcal{E}$.

Take $\underline{S} = \{S_1, \dots, S_r\}$ some collection of simple objects in $\mathcal{E}$.

$Q = (Q_0, Q_1)$ the Ext-quiver

$$\mathcal{M}_{\Pi_Q} \xleftarrow{i_{nil}} \mathbb{N}^{Q_0} \xrightarrow{i_{\underline{S}}} \mathcal{M}_{\mathcal{E}}$$

$i_{\underline{S}}^! \Psi_{\mathcal{E}} \simeq i_{nil}^! \Psi_{\Pi_Q}$ is indeed an isomorphism by the PBW theorem for preprojective algebras (coming from dimensional reduction).