Lecture 1 : brief summary

I - Constructible derived category

$\mathcal{D}_c(X, \mathbb{Q})$ $\quad X$ $\mathbb{C}$-alg variety

$\mathrm{Perv}(X)$

If $\mathcal{M}$ is a monoid in the category of $\mathbb{C}$-schemes,

$(\mathcal{D}_c^+(\mathcal{M}), \boxtimes)$ monoidal structure

If $\oplus : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ finite, $(\mathrm{Perv}(\mathcal{M}), \boxtimes)$ has a monoidal structure

Monoidal functors can be used to transfer algebra objects to other categories

All classical constructions work well: Free associative / Lie algebras, ideals, enveloping algebras, PBW theorem.

Lecture 2: II - 2-Calabi-Yau categories and moduli stacks

III - 2d CoHA structure and BPS algebra

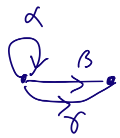
IV - Critical CoHA and dimensional reduction

II - 2-Calabi–Yau categories and their moduli stacks

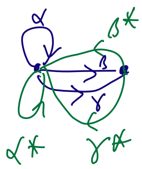
Note: I am not a derived algebraic geometer: I have a pedestrian approach (when derived objects appear).

① Examples

ⓐ Preprojective algebras. $Q = (Q_0, Q_1)$ quiver, Q_0 vertices, Q_1 arrows

e.g. 

$\overline{Q} = (Q_0, Q_1 \sqcup Q_1^{op})$ double quiver



$$\rho = \sum_{\alpha \in Q_1} (\alpha\alpha^* - \alpha^*\alpha) \in \mathbb{C}\overline{Q} \text{ path algebra of } \overline{Q}$$

$$\Pi_Q := \mathbb{C}\overline{Q} / \langle\langle \rho \rangle\rangle$$

Thm (Crawley-Boevey) Π_Q is a 2CY algebra, if Q not Dynkin ADE

Rk: If Q is Dynkin ADE, work with Ginzburg dg-algebra instead.

Stack of objects, $d \in \mathbb{N}^{Q_0}$: $\quad X_{Q,d} = \bigoplus_{\alpha \in Q_1} \mathrm{Hom}(\mathbb{C}^{d_{s(\alpha)}}, \mathbb{C}^{d_{t(\alpha)}})$ ↺

$$X_{\overline{Q},d} \cong T^* X_{Q,d} \circlearrowleft GL_d = \prod_{i \in Q_0} GL_{d_i}$$

Hamiltonian

$$\mu_d : T^*X_{Q,d} \longrightarrow \mathfrak{gl}_d \quad \text{moment map}$$
$$(x_\alpha, x_{\alpha^*})_{\alpha \in Q_1} \mapsto \sum_{\alpha \in Q_1} [x_\alpha, x_{\alpha^*}]$$

$$\mathfrak{M}_{\Pi_{Q,d}} := \left[\mu_d^{-1}(0) / GL_d \right] \quad \text{quotient stack}$$

$$\begin{tikzcd} \mathfrak{M}_{\Pi_{Q,d}} \arrow[d, "JH"'] \\ \mathcal{M}_{\Pi_{Q,d}} \end{tikzcd}$$

$$\mathcal{M}_{\Pi_{Q,d}} := \mu_d^{-1}(0) /\!/ GL_d \quad \text{affine GIT quotient.}$$

(a') Multiplicative versions of preprojective algebras

(b) Sheaves on symplectic surfaces

S K3 or Abelian surface

or $S = T^*C$ for C smooth projective curve.

H polarization

$\mathrm{Coh}^{H\text{-}ss}_{p(t)}(S)$ semistable sheaves on S w/ normalized Hilbert polynomial $p(t)$.

$\mathfrak{M}^{H\text{-}ss}_{p(t)}(S)$ Classical constructions using Quot-schemes.

$$\begin{tikzcd} \mathfrak{M}^{H\text{-}ss}_{p(t)}(S) \arrow[d, "JH"] \\ \mathcal{M}^{H\text{-}ss}_{p(t)}(S) \end{tikzcd}$$

ⓒ S Riemann surface, of genus g

$$\pi_1(S,x) \simeq \left\{ x_i, y_i : 1 \leq i \leq g \;\middle|\; \prod_{i=1}^{g} x_i y_i x_i^{-1} y_i^{-1} = 1 \right\}$$

ordered product

<u>Thm</u> (Davison) Let $g \geq 1$. $\mathrm{Rep}\, \pi_1(S,x)$ is 2CY.

Construction of moduli stacks and spaces is a particular case of multiplicative preprojective algebra.

Use the multiplicative moment map

$$\mu_n : GL_n^{2g} \longrightarrow GL_n \qquad n \geq 1$$

$$(M_i, N_i)_{1 \leq i \leq g} \longmapsto \prod_{i=1}^{g} M_i N_i M_i^{-1} N_i^{-1}$$

$$\mathfrak{M}_{g,n} = \left[\mu_n^{-1}(\mathrm{Id}_n) / GL_n \right]$$

$$\downarrow \mathrm{JH}$$

$$\mathcal{M}_{g,n} = \mu_n^{-1}(\mathrm{Id}_n) /\!/ GL_n \quad .$$

② 2-Calabi-Yau categories

We put all categories as above under the umbrella of what we call

2-Calabi-Yau Abelian categories.

$\mathcal{D}$ = "ambient" pretriangulated dg-category

$\mathfrak{M}_{\mathcal{D}}$ = derived moduli stack of objects in $\mathcal{E}$

$\mathcal{E} \subset H^0(\mathcal{D})$ Abelian category s.t.

$$\mathfrak{M}_{\mathcal{E}} \underset{\substack{\text{1-Artin} \\ \text{substack}}}{\overset{\text{open}}{\subset}} \mathfrak{M}_{\mathcal{D}} .$$

2 Calabi-Yau structure

$\forall x_1, \dots, x_r \in \mathcal{M}_{\mathcal{E}}$, corresponding to simple objects $\mathcal{F}_1, \dots, \mathcal{F}_r \in \mathcal{E}$, the full dg-subcategory $\mathcal{D}'$ of $\mathcal{D}$ generated by $\mathcal{F}_1, \dots, \mathcal{F}_r$ has a right 2-Calabi-Yau structure [Brav-Dyckerhoff].

Roughly, this means that we have bi-functorial isomorphisms

$$\mathrm{Hom}_{H^0(\mathcal{D})}(\mathcal{F}, \mathcal{G}[i]) \cong \mathrm{Hom}_{H^0(\mathcal{D})}(\mathcal{G}, \mathcal{F}[2-i])^*$$

$\forall \mathcal{F}, \mathcal{G} \in \mathcal{D}'$

Good moduli space

We assume that $\mathfrak{M}_{\mathcal{E}}$ has a good moduli space in the sense of Alper-Rydh-Hall :

$$JH_{\mathcal{E}} : \mathfrak{M}_{\mathcal{E}} \longrightarrow \underbrace{\mathcal{M}_{\mathcal{E}}}$$

usually algebraic space

Assume: finite type, separated $\mathbb{C}$-scheme.

In particular, $JH_{\mathcal{E}}$ is universal among maps to an algebraic space.

$\oplus$ - morphism

$\oplus : \mathfrak{M}_{\mathcal{E}} \times \mathfrak{M}_{\mathcal{E}} \longrightarrow \mathfrak{M}_{\mathcal{E}}$ directsum, induces (by universality of $JH_{\mathcal{E}}$) $\oplus : \mathcal{M}_{\mathcal{E}} \times \mathcal{M}_{\mathcal{E}} \longrightarrow \mathcal{M}_{\mathcal{E}}$. finite map

RHom complex: If $X = \operatorname{Spec}(\overset{k\text{-algebra}}{A})$, X-points of $\mathfrak{M}_{\mathcal{D}}$ are pseudo-perfect $\mathcal{D} \underset{\mathbb{C}}{\otimes} A$-module N.

For N, N' such points, $\mathrm{RHom}_{\mathcal{E} \underset{\mathbb{C}}{\otimes} A}(N, N')$ is a dg-A module.

$\rightsquigarrow$ defines the RHom complex on $\mathfrak{M}_{\mathcal{D}}^{\times 2}$ and, by restriction, on $\mathfrak{M}_{\mathcal{E}}^{\times 2}$.

$$C := \mathrm{RHom}[1]$$

Stack of short exact sequences

$$\mathcal{E}xact_{\mathcal{E}} \simeq \mathrm{Tot}(C).$$

$$\begin{tikzcd}
\mathfrak{M}_{\mathcal{E}} \times \mathfrak{M}_{\mathcal{E}} \arrow[d, "JH_{\mathcal{E}} \times JH_{\mathcal{E}}"'] & \mathcal{E}xact_{\mathcal{E}} \arrow[l, "q"', "\text{quasi-smooth}"] \arrow[r, "p", "\text{proper}"'] \arrow[d, phantom, "\circlearrowright"] & \mathfrak{M}_{\mathcal{E}} \arrow[d, "JH_{\mathcal{E}}"] \\
\mathcal{M}_{\mathcal{E}} \times \mathcal{M}_{\mathcal{E}} \arrow[rr, "\oplus"'] & & \mathcal{M}_{\mathcal{E}}
\end{tikzcd}$$

The local neighbourhood theorem

Ext quivers

dg $\mathcal{D} \supset \mathcal{E}$ finite length Abelian category

$S = \{S_1, \dots, S_r\}$ pairwise non-iso simple objects of $\mathcal{E}$.

$\overline{Q} = (Q_0, Q_1)$ Ext-quiver of S:

$Q_0 = S = \{S_1, \dots, S_r\}$

and $\# \{ i \to j \} := \dim \operatorname{Hom}_{H^0(\mathcal{D})} (S_i, S_j[1])$

Local neighbourhood theorem

$$\begin{array}{ccccc} \mathcal{M}_{\mathcal{E}} & \longleftarrow & \mathbb{N}^{Q_0} & \longrightarrow & \mathcal{M}_{\Pi_Q} \\ S_{\underline{m}} := \bigoplus_{i=1}^{r} S_i^{m_i} & \longleftarrow\!\shortmid & \underline{m} & \longmapsto & 0_{\underline{m}} \end{array}$$

Upshot: locally, the map $JH : \mathfrak{M}_{\mathcal{E}} \to \mathcal{M}_{\mathcal{E}}$ looks like the map $JH : \mathfrak{M}_{\Pi_Q} \to \mathcal{M}_{\Pi_Q}$ for Q quiver.

More precisely

Thm: [Davison] $\mathcal{E}, \mathcal{D}, \underline{S}$ as above

$\overline{Q}$ Ext-quiver of $\underline{S}$

$\forall \underline{m} \in \mathbb{N}^{Q_0}$, $\exists\, U \circlearrowleft GL_{\underline{m}}$, and a diagram with Cartesian squares and étale horizontal maps.

$$\begin{tikzcd}
\left(\mathfrak{M}_{\Pi_{Q_{\underline{m}}}}, \underline{O}_m\right) \arrow[d, "JH_{\Pi_Q}"'] & \left[U/GL_{\underline{m}}\right] \arrow[l] \arrow[r] \arrow[d] & \left(\mathfrak{M}_{\mathcal{E}}, S_{\underline{m}}\right) \arrow[d, "JH_{\mathcal{E}}"] \\
\left(\mathcal{M}_{\Pi_{Q_{\underline{n}}}}, \underline{O}_m\right) & U/\!/GL_{m} \arrow[l] \arrow[r] & \left(\mathcal{M}_{\mathcal{E}}, S_{\underline{m}}\right)
\end{tikzcd}$$

In addition, we have compatibility with the RHom-complexes:

$$RHom_{\Pi_{Q_{\underline{n}}}}\Big|_{\left(JH_{\Pi_Q} \times JH_{\Pi_Q}\right)^{-1}\left(O_{\underline{m}} \times O_{\underline{n}}\right)} \cong RHom_{\Pi_{Q_{\underline{n}}}}\Big|_{\left(JH_{\mathcal{E}} \times JH_{\mathcal{E}}\right)^{-1}\left(\{x, y\}\right)}$$

III – 2d Cohomological Hall algebra structure and BPS algebra

① 2d CoHA structure, from Kapranov–Vasserot construction.

$\mathcal{E}$ 2CY Abelian category

$$\begin{tikzcd} \mathfrak{M}_{\mathcal{E}} \arrow[d, "\mathrm{JH}_{\mathcal{E}}"] \\ \mathcal{M}_{\mathcal{E}} \end{tikzcd}$$

$\mathfrak{M}_{\mathcal{E}}$ stack of objects

$\mathrm{JH}_{\mathcal{E}}$ good moduli space

$\mathcal{C} = R\mathcal{H}om[1]$ 3-term complex of vector bundles over $\mathfrak{M}_{\mathcal{E}} \times \mathfrak{M}_{\mathcal{E}}$ (intrinsically given by derived geometry of $\mathfrak{M}_{\mathcal{E}}$).

$\mathfrak{Exact}_{\mathcal{E}} = \mathrm{Tot}(\mathcal{C})$ stack of short exact sequences

$$\mathcal{C} = \left(\mathcal{C}^{-1} \xrightarrow{d^{-1}} \mathcal{C}^{0} \xrightarrow{d^{0}} \mathcal{C}^{1}\right)$$

$$\begin{tikzcd} \mathfrak{M}_{\mathcal{E}} \times \mathfrak{M}_{\mathcal{E}} \arrow[d, "\mathrm{JH}_{\mathcal{E}} \times \mathrm{JH}_{\mathcal{E}}"'] & \mathfrak{Exact}_{\mathcal{E}} \arrow[l, "q"'] \arrow[r, "p \text{ proper}"] \arrow[d, phantom, "\circlearrowleft"] & \mathfrak{M}_{\mathcal{E}} \arrow[d, "\mathrm{JH}_{\mathcal{E}}"] \\ \mathcal{M}_{\mathcal{E}} \arrow[rr, "\oplus"'] & & \mathcal{M}_{\mathcal{E}} \end{tikzcd}$$

The underlying constructible complex of the sheafified CoHA is

$$\mathcal{A}_{\mathcal{E}} := \mathrm{JH}_{*} \mathbb{D}\mathbb{Q}^{\mathrm{vir}}_{\mathfrak{M}_{\mathcal{E}}}$$

$\mathrm{vir} = \mathrm{rk}\, \mathbb{T}_{\mathfrak{M}_{\mathcal{E}}}$ rk of the tangent complex of the natural derived enhancement of $\mathfrak{M}_{\mathcal{E}}$.

locally constant function on $\mathfrak{M}_{\mathcal{E}}$.

$\rightsquigarrow$ most natural shift. If $\mathfrak{M}_{\mathcal{E}}$ were smooth of the expected virtual dimension, it makes $\mathbb{Q}^{\mathrm{vir}}_{\mathfrak{M}_{\mathcal{E}}}$ perverse.

q is "quasi-smooth". We can build the pullback by q in a very explicit way using the diagram

$$\begin{tikzcd}
 & & \mathcal{E}xact_{\mathcal{E}} \simeq t_0(\mathrm{Tot}\,C) \arrow[r, "f"] \arrow[d, "g"'] \arrow[dll, "q"'] & C^0/C^{-1} \arrow[d, "s_0"', "\text{zero section}"] \\
\mathfrak{M}_{\mathcal{E}} \times \mathfrak{M}_{\mathcal{E}} & \mathrm{Tot}(C^{-1} \xrightarrow{d^{-1}} C^0) \arrow[l, "\pi"'] \arrow[r, phantom, "="] & C^0/C^{-1} \arrow[r, bend right, "\pi^* d^0 =: s_d"'] & \pi^* C^1 \arrow[l]
\end{tikzcd}$$

vector bundle stack, hence smooth

p is proper: pushforward by p.

Altogether:

$$(JH_{\mathcal{E}} \times JH_{\mathcal{E}})_* \mathbb{D}\mathbb{Q}^{vir}_{\mathfrak{M}_{\mathcal{E}} \times \mathfrak{M}_{\mathcal{E}}} \xrightarrow{\quad m \quad} \mathbb{D}\mathbb{Q}^{vir}_{\mathfrak{M}_{\mathcal{E}}}$$

<u>Thm</u>: $\mathcal{A}_{\mathcal{E}} := \left(JH_* \mathbb{D}\mathbb{Q}^{vir}_{\mathfrak{M}_{\mathcal{E}}},\ m \right) \in \mathcal{D}^+_c(\mathcal{M}_{\mathcal{E}}, \mathbb{Q})$ is an associative algebra object.

② BPS associative algebra

Proposition (Davison)

$$ {}^{p}\mathcal{H}^{i}(\mathcal{A}_{\mathcal{C}}) = 0 \quad \text{for} \quad i<0 $$

i.e. $\mathcal{A}_{\mathcal{C}}$ is concentrated in nonnegative perverse degrees.

Proof: $\mathcal{C} = \mathrm{Rep}\, \Pi_Q$: relies on critical CoHA of the triple quiver w/ potential + dimensional reduction (Davison)

$\mathcal{C}$ general: local neighbourhood theorem.

since being in ≥ 0 perverse degrees can be checked étale locally on $\mathcal{M}_{\mathcal{C}}$. □

Definition: $\mathcal{BPS}_{\mathcal{C},\mathrm{Alg}} := {}^{p}\mathcal{H}^{0}(\mathcal{A}_{\mathcal{C}})$.

By abstract nonsense (adjunctions), we have

$$ {}^{p}\tau_{\leq 0}\mathcal{A}_{\mathcal{C}} \simeq \mathcal{BPS}_{\mathcal{C},\mathrm{Alg}} \longrightarrow \mathcal{A}_{\mathcal{C}} $$

and $\quad m : \mathcal{A}_{\mathcal{C}} \boxdot \mathcal{A}_{\mathcal{C}} \longrightarrow \mathcal{A}_{\mathcal{C}} \quad$ induces

$$ m : \mathcal{BPS}_{\mathcal{C},\mathrm{Alg}}^{\boxdot 2} \longrightarrow \mathcal{BPS}_{\mathcal{C},\mathrm{Alg}}. $$

Corollary: $(\mathcal{BPS}_{\mathcal{C},\mathrm{Alg}}, m) \in \mathrm{Perv}(\mathcal{M}_{\mathcal{C}})$ is an associative algebra object.

IV- A glimpse into CoHAs of quivers with potential (3d)

following Kontsevich–Soibelman, Davison–Meinhardt

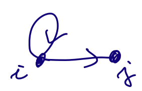
A few essential properties come from the description of $\mathcal{A}_\mathcal{E}$ as CoHA of a quiver w/ potential.

1. $\mathcal{A}_\mathcal{E}$ is a semisimple complex
2. PBW for $\mathcal{A}_\mathcal{E}$
3. $\mathcal{A}_\mathcal{E}$ is concentrated in nonnegative perverse degrees

1 The critical CoHA (Kontsevich–Soibelman)

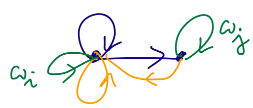
Defined for any quiver with potential.

We concentrate on tripled quivers with their canonical potential.

Q quiver 

$\bar{Q}$ double 

$\bar{Q} = (Q_0, \bar{Q}_1)$ $\bar{Q}_1 = Q_1 \sqcup Q_1^*$

$\tilde{Q}$ triple 

$\tilde{Q} = (Q_0, \tilde{Q}_1)$ $\tilde{Q}_1 = \bar{Q}_1 \sqcup Q_0$, $Q_0 = \{\omega_i : i \in Q_0\}$

canonical potential $W = \left(\sum_{i\in Q_0} \omega_i\right)\left(\sum_{\alpha\in Q_1} [\alpha,\alpha^*]\right)$.

Tr W gives a regular function

$\mathrm{Tr}W : \mathfrak{M}_{\tilde{Q},d} \to \mathbb{A}^1$ $\forall d \in \mathbb{N}^{Q_0}$.

Donaldson–Thomas sheaf

vanishing cycle sheaf shifted so that it preserves perversity.

$$\mathcal{DT}_{W,d} := \varphi^p_{\mathrm{Tr}\,W} \, \mathbb{Q}_{\mathfrak{M}_{\tilde{Q},d}}[\dim \mathfrak{M}_{\tilde{Q},d}]$$
$$\in \mathrm{Perv}(\mathfrak{M}_{\tilde{Q},d})$$

$$JH_{\tilde{Q}} : \mathfrak{M}_{\tilde{Q}} \longrightarrow \mathcal{M}_{\tilde{Q}}$$

$JH_{\tilde{Q}}$ is approximable by proper maps

$$\Rightarrow \mathcal{A}_{\tilde{Q},W} := JH_{\tilde{Q}*} \mathcal{DT}_{W,d} \in {}^p\mathcal{D}^{\geq 1}(\mathcal{M}_{\tilde{Q}}) .$$

Theorem (Davison–Meinhardt)

* $\mathcal{H}\mathcal{A}_{\tilde{Q},W} := \bigoplus_{i \geq 1} ({}^p\mathcal{H}^i \mathcal{A}_{\tilde{Q},W})[-i]$ has an induced algebra structure
* It is commutative ($\Rightarrow$ the Lie bracket is trivial).

② The BPS Lie algebra and PBW theorem

BPS Lie algebra

$\Rightarrow \quad {}^{p}\mathcal{H}^{1} \mathcal{A}_{\tilde{Q},W} =: \mathcal{BPS}_{\tilde{Q},W,\mathrm{Lie}}$ is a Lie algebra object in $\mathrm{Perv}(\mathcal{M}_{\tilde{Q}})$.

PBW theorem

$$\mathrm{Sym}\Big(\underbrace{\mathcal{BPS}_{\tilde{Q},W} \otimes [-1]}_{\text{sits in perverse degree one.}} \otimes H^*_{\mathbb{C}^*}(pt)\Big) \xrightarrow{\sim} \mathcal{A}_{\tilde{Q},W}$$

iso in $\mathcal{D}^+_c(\mathcal{M}_{\tilde{Q}})$.

Support theorem

$$\begin{tikzcd} \mathcal{M}_{\overline{Q}} \arrow[d, hook, "\mathrm{id}\times 0"'] \arrow[dr, hook, "i'"] & \\ \mathcal{M}_{\overline{Q}} \times \mathbb{C}^{Q_0} \arrow[r, hook, "i"] & \mathcal{M}_{\tilde{Q},W} \end{tikzcd}$$

request the additional loops to act by a scalar.

Theorem: * $\mathrm{Supp}\ \mathcal{BPS}_{\tilde{Q},W} \subset \mathcal{M}_{\overline{Q}} \times \mathbb{C}^{Q_0}$.

* $\mathcal{BPS}_{\tilde{Q},W} \cong i_*\Big(\underbrace{i'^* \mathcal{BPS}_{\tilde{Q},W}}_{\text{supported of } \mathcal{M}_{\Pi_Q} \hookrightarrow \mathcal{M}_{\overline{Q}}} \boxtimes \mathbb{Q}_{\mathbb{C}^{Q_0}}\Big)$

③ Dimensional reduction for the triple quiver w/ potential

Dimensional reduction

forget the loops:

$$\begin{tikzcd} \mathfrak{M}_{\tilde{Q}} \arrow[r, "\pi"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \mathfrak{M}_{\overline{Q}} \arrow[d] \\ \mathcal{M}_{\tilde{Q}} \arrow[r, "\pi"'] & \mathcal{M}_{\overline{Q}} \end{tikzcd}$$

Thm (Davison) * $\pi_* \mathcal{DT}_{\tilde{Q},W} \cong JH_* \mathbb{D}\mathbb{Q}^{vir}_{\mathfrak{M}_{\Pi_Q}}$ [up to some appropriate shifts]

* Both sides come with their multiplication.
they coincide

Consequence for the BPS sheaf and the 2D CoHA

$$\mathcal{BPS}^{3d}_{\Pi_Q, Lie} := \pi_* \mathcal{BPS}_{\tilde{Q},W}[-1] \qquad \in \mathrm{Perv}(\mathcal{M}_{\Pi_Q})$$

$$\cong i^* \mathcal{BPS}_{\tilde{Q},W}[-1].$$

2d PBW theorem

(*) $$\mathrm{Sym}_{\boxdot}\left(\mathcal{BPS}^{3d}_{\Pi_Q, Lie} \otimes H^*_{\mathbb{C}^*}(pt)\right) \xrightarrow{\sim} \mathcal{A}_{\Pi_Q}.$$

$$\begin{tikzcd} \mathrm{Sym}_{\boxdot}\left(\mathcal{BPS}^{3d}_{\Pi_Q, Lie} \otimes H^*_{\mathbb{C}^*}(pt)\right) \arrow[r, "\sim"] & \mathcal{A}_{\Pi_Q} \\ \mathcal{BPS}^{3d}_{\Pi_Q, Lie} \arrow[u] \arrow[r, "\text{Lie alg}"] & \mathcal{BPS}_{\Pi_Q, Alg} \arrow[u] \end{tikzcd}$$

+ ${}^{p}\mathcal{H}^0$ of (*) gives $\mathrm{Sym}_{\boxdot}\left(\mathcal{BPS}^{3d}_{\Pi_Q, Lie}\right) \xrightarrow{\sim} \mathcal{BPS}_{\Pi_Q, Alg}$.

and so $\boxed{\mathcal{BPS}_{\Pi_Q, Alg} \cong U\left(\mathcal{BPS}^{3d}_{\Pi_Q, Lie}\right)}$ as algebra objects.