The quantum group of a quiver

Lusztig, Bozec: algebra structure, canonical basis, semicanonical basis

not investigated: geometric study of bialgebra structure

I The quantum group of a quiver

(approach following Lusztig, Bozec)

$Q = (Q_0, Q_1)$ quiver

$$\begin{tikzcd} \bullet \arrow[loop left] \arrow[r] & \bullet \end{tikzcd}$$

loops, multiple edges are allowed

Lie algebra / quantum group associated with Q

$\rightsquigarrow$ generalises quantum groups of Kac-Moody type (obtained when Q has no loops)

$$\mathfrak{g}_Q = \mathfrak{n}_Q^- \oplus \mathfrak{h} \oplus \mathfrak{n}_Q^+$$

$$U(\mathfrak{g}_Q) = U^{\mathbb{Z}}(\mathfrak{n}_Q^-) \otimes U^{\mathbb{Z}}(\mathfrak{h}) \otimes U^{\mathbb{Z}}(\mathfrak{n}_Q^+)$$

$$U_q(\mathfrak{g}_Q) = U_q^{\mathbb{Z}}(\mathfrak{n}_Q^-) \otimes U_q^{\mathbb{Z}}(\mathfrak{h}) \otimes U_q^{\mathbb{Z}}(\mathfrak{n}_Q^+)$$

$\mathbb{Z}$: integral part.

Today: concentrate on the positive parts $\mathfrak{n}_Q^+$, $U(\mathfrak{n}_Q^+)$, $U_q(\mathfrak{n}_Q^+)$.

As a ~~b~~ialgebra, $U_q(\mathfrak{n}_Q^+)$ is generated by subalgebras

$$U_q(\mathfrak{n}_Q^+)_i \subset U_q(\mathfrak{n}_Q^+)$$

$$U_q(\mathfrak{n}_Q^+)_i \cong \begin{cases} \mathbb{Z}[q][x] & \text{if } i \text{ has no loops} \\ \Lambda_{\mathbb{Z}[q]} & \text{MacDonald ring of symmetric functions if } i \text{ has 1 loop} \\ \Lambda^{nc}_{\mathbb{Z}[q]} ? & \text{ring of noncommutative symmetric functions if } i \text{ has} \geq 2 \text{ loops.} \end{cases}$$

$\mathbb{C}(q)[\tilde{e}_{i,n}, n \geq 1] =$

$\mathbb{C}(q)\langle \tilde{e}_{i,n} ; n \geq 1 \rangle =$

these subalgebras interact via Serre relations

if quantum group, q-deformation of noncommutative symm functions?

q-deformed nc symmetric functions?

In explicit terms,

Q_0^{real} = vertices without loops

Q_0^{im} = vertices with loops

$$I_\infty = (Q_0^{real} \times \{1\}) \sqcup (Q_0^{im} \times \mathbb{Z}_{\geq 1}) \quad \text{. set of simple positive roots}$$

bilinear pairing:

$$\mathbb{Z}^{Q_0} \times \mathbb{Z}^{Q_0} \longrightarrow \mathbb{Z}$$

$$(d, e) \longmapsto 2\sum_{i \in Q_0} d_i e_i - \sum_{i \xrightarrow{\alpha} j \in Q_1} (d_i e_j + e_i d_j)$$

"symmetrised Euler form of Q"

$$\mathbb{Z}^{(I_\infty)} \longrightarrow \mathbb{Z}^{Q_0}$$

$$(i', n) \longmapsto n i' ;$$

bilinear form $(-,-)$ on $\mathbb{Z}^{(I_\infty)}$ by pullback.

$U_q(\mathfrak{n}_Q^+)$ is generated by $e_{(i',n)}$, $(i',n) \in I_\infty$, with the relations

① $[e_i, e_j] = 0$ if $(i,j) = 0$

② $\displaystyle\sum_{k+l=1-(i,j)} \binom{1-(i,j)}{k}_q e_i^k e_j e_i^l$ if $i \in Q_0^{real}$

quantum binomial coefficient

$q=1$: make $q \to 1$, i.e. replace $\binom{v}{u}_q$ by $\binom{v}{u}$

$$\binom{n}{k}_q = \frac{[n]_q!}{[k]_q!\,[n-k]_q!}$$

$$[n]_q! = \prod_{l=1}^{n} [l]_q$$

$1+q^2+\dots+q^{2l-2}$

$$[l]_q = \frac{q^l - q^{-l}}{q-q^{-1}} = \begin{cases} q^{-l+1}\cdot\dfrac{q^{2l}-1}{q^2-1} \\ q^{-l+1} + q^{-l+3} + \dots + q^{l-3} + q^{l-1}. \end{cases}$$

<u>bialgebra structure</u> $\Delta : U_q(\mathfrak{n}_Q^+) \to U_q(\mathfrak{n}_Q^+) \otimes_{\xi_q} U_q(\mathfrak{n}_Q^+)$

$\Delta e_i = e_i \otimes 1 + 1 \otimes e_i.$

It is a ξ_q- twisted bialgebra structure:

$$\xi_q(d,e) = q^{(d,e)}$$

$U_q(n_Q^+) \otimes U_q(n_Q^+)$ is naturally a $M \times M$ - graded algebra, $M = \mathbb{N}^{Q_0}$

twisted product: $(x \otimes y)(z \otimes t) = \xi_q(|y|,|z|)\big((xy)\otimes(zt)\big)$
for $x, y, z, t \in U_q(n_Q^+)$ homogeneous.

<u>integral form</u> $U_q^{\mathbb{Z}}(n_Q^+)$ $\mathbb{Z}[q^{\pm 1}]$ - subalgebra of $U_q(n_Q^+)$ generated by e_i, $i \in I_\infty$

$$e_i^{(n)} := \frac{e_i^n}{[n]_q!}, \quad n \geq 1.$$

Geometric constructions often gives $U_{q^{-1}}(n_Q^+)$ or $U_{-1}(n_Q^+)$ when we would like to obtain $U_q(n_Q^+)$, $U(n_Q^+)$: we need to be able to twist.

$\boxed{\text{II}}$. Twisting algebras

- All algebras associated with Q are $\mathbb{N}^{Q_0}$ - graded.

<u>More generally</u>

- R a ring
- M a monoid
- A a M-graded R-algebra $\rightsquigarrow$ product $\ast : A \otimes A \to A$. R-linear (+ associativity condition)

$$A = \bigoplus_{m \in M} A_m$$

Twist of the product

$\psi : M \times M \to R$ multiplicative bilinear form.

in part., $\psi(m+n, p) = \psi(m,p)\psi(n,p)$

$x \ast_\psi y = \psi(|x|, |y|)(x \ast y)$ defines a new associative product. B^ψ associated algebra

- **Twisting twisted bialgebras** "ξ-twisted bialgebra"

$m : B \otimes B \to B$ multiplication

$\Delta : B \longrightarrow B \overset{\xi}{\otimes} B$ algebra map

$$(x \otimes y)(z \otimes t) = \xi(|y|, |z|)(xz \otimes yt)$$

if $\psi : M \times M \to R^\times$ mult. bilin. form; takes invertible values

* B^{Ψ} twisted algebra.

* twist comultiplication

$$\Delta = \sum_{u,v \in M} \Delta_{u,v}$$

$$\Delta^{\Psi} = \sum_{u,v \in M} \frac{1}{\Psi(u,v)} \Delta_{u,v}$$

$\Delta^{\Psi} : B \longrightarrow B \overset{\xi'}{\otimes} B$ algebra morphism, where

$$\xi'(u,v) = \frac{\Psi(v,u)}{\Psi(u,v)} \xi(u,v)$$

Modification of the ξ-twist by an algebra automorphism

- B ξ-twisted bialgebra, $\xi : M \times M \longrightarrow R$
- $f : B \longrightarrow B$ ring automorphism preserving R
- $f \circ \xi : M \times M \longrightarrow R$

$f \otimes f : A \overset{\xi}{\otimes} A \longrightarrow A \overset{f\circ\xi}{\otimes} A$ is an algebra isomorphism (not of R-algebras)

$(f \otimes f) \circ \Delta \circ f^{-1}$ makes A a $f \circ \xi$-twisted bialgebra.

Opposite parameter

$U_q^{\mathbb{Z}}(\mathfrak{n}_Q^+)$ and $U_{-q}^{\mathbb{Z}}(\mathfrak{n}_Q^+)$ are two distinct $\mathbb{Z}[q,q^{-1}]$-algebras.

Prop: $\varphi : \mathbb{N}^{Q_0} \times \mathbb{N}^{Q_0} \to \mathbb{Z}$ such that

$$\varphi(d,e)\varphi(e,d) = (-1)^{(d,e)} \quad \forall d,e \in \mathbb{N}^{Q_0}.$$

Then, $U_q^{\mathbb{Z}}(\mathfrak{n}_Q^+)^{\varphi} \cong U_{-q}(\mathfrak{n}_Q^+)$.

Proof:
$$\begin{pmatrix} n \\ k \end{pmatrix}_{-q} = \begin{cases} \begin{pmatrix} n \\ k \end{pmatrix}_q & \text{if } \begin{cases} n,k \equiv 0 \, [2] \\ \text{or} \\ n \text{ is odd} \end{cases} \\ -\begin{pmatrix} n \\ k \end{pmatrix}_q & \text{if } \begin{cases} n \equiv 0 \, [2] \\ \text{and} \\ k \equiv 1 \, [2] \end{cases} \end{cases}$$

\+ comparison of Serre relations

$\square$

In geometric constructions, the generators are often not primitive (for the natural coproduct)

→ related to the rep. theory of Hecke algebras of type A at $q=0$

III - New generators for $U_q(\mathfrak{g}_a)$

Ⓐ Non-commutative symmetric functions (Gelfand, Krob, Lascoux, Leclerc, Retakh, Thibon)

$\Lambda = \mathbb{C}\langle \Lambda_1, \Lambda_2, \dots \rangle$ free algebra generated by infinitely many indeterminates

Λ_k = " elementary symmetric functions" ; $\Lambda_0 := 1$

* commutative world : partitions
* non-commutative world : compositions

$$\text{compositions of } n = \overbrace{(n_1, \dots, n_k)}^{I} \quad \text{s.t.} \quad \sum_{j=1}^{k} n_j = n .$$

not necessarily decreasing

compositions of n index various bases of Λ,

for example $\Lambda^I = \Lambda_{i_1} \cdots \Lambda_{i_k} \in \Lambda$.

It is convenient to use generating series .

$$\lambda(t) = \sum_{k \geq 0} t^k \Lambda_k .$$

* complete homogeneous symmetric functions :

$$\sigma(t) := \sum_{k \geq 0} t^k S_k = \lambda(-t)^{-1}$$

* power sum symmetric functions of the first kind Ψ_k

$$\Psi(t) := \sum_{k \geq 1} t^{k-1} \Psi_k$$

$$\frac{d}{dt} \sigma(t) = \sigma(t) \Psi(t) .$$

* power sum symmetric functions of the second kind Φ_k

$$\sigma(t) = \exp\left(\sum_{k \geq 1} t^k \frac{\Phi_k}{k} \right).$$

Proposition: Λ is freely generated by any of the families $\Lambda_k, S_k, \Psi_k, \Phi_k$

Comultiplication:

$$\Delta \Psi_k = \Psi_k \otimes 1 + 1 \otimes \Psi_k$$

Prop: $\Delta S_k = \sum_{i=0}^{k} S_i \otimes S_{k-i}$

$$\Delta \Lambda_k = \sum_{i=0}^{k} \Lambda_i \otimes \Lambda_{k-i}$$

Lie algebras: The Lie subalgebras of Λ generated by Φ_k and Ψ_k coincide.

$\Lambda^I = \Lambda_{m_1} \cdots \Lambda_{m_r}$ I composition,

$S^I = S_{m_1} \cdots S_{m_r}$ products

$\phi^I = \phi_{m_1} \cdots \phi_{m_r}$

$\psi^I = \psi_{m_1} \cdots \psi_{m_r}$

$\mathcal{I}$- bases of Λ

base change matrices between ψ and S:

$$n S_n = \begin{vmatrix} \psi_1 & \cdots & \psi_{n-1} & \boxed{\psi_n} \\ -1 & \psi_1 \cdots & \psi_{n-2} & \psi_{n-1} \\ 0 & -2\,\psi_1 & \psi_{n-3} & \psi_{n-2} \\ & & & \\ 0 & \cdots & -n+1 & \psi_1 \end{vmatrix}$$

quasi determinant.

Corollary of this formula

$$S_r = \psi_r + \sum_{\substack{s \geq 2 \\ r_1 + \cdots + r_s = r \\ r_i \geq 1}} a_{r_1 \cdots r_s} \prod \psi_{r_j}$$

$\in \mathbb{Q}$

q-deformed version

* $\Lambda_q = \mathbb{C}(q)\langle \Lambda_1, \Lambda_2, \dots \rangle$ same definition but over the field $\mathbb{C}(q)$.

$\mathbb{N}$-graded algebra for $\deg \Lambda_i = i$

* comultiplication $\Delta : \Lambda_q \longrightarrow \Lambda_q \otimes_{\xi_q} \Lambda_q$ twist

$$\Delta \Psi_k = \Psi_k \otimes 1 + 1 \otimes \Psi_k$$

$$\xi_q(r,s) = q^{(g-1)rs}$$

Conjecture (probably not too difficult) There exists elements of Λ_q, α_r, $r \geq 1$ such that

$$\Delta \alpha_r = \sum_{s+t=r} q^{(g-1)st} \alpha_s \otimes \alpha_t \qquad \text{for some } g \geq 1$$

e.g. $\Delta \alpha_1 = \alpha_1 \otimes 1 + 1 \otimes \alpha_1 \rightarrow \alpha_1 = \Psi_1$

$$\Delta \alpha_2 = \alpha_2 \otimes 1 + 1 \otimes \alpha_2 + q^{g-1} \alpha_1 \otimes \alpha_1$$

$$\Delta \alpha_1^2 = \alpha_1^2 \otimes 1 + 1 \otimes \alpha_1^2 + (1 + q^{g-1})\, \alpha_1 \otimes \alpha_1$$

$$\rightsquigarrow \alpha_2 = \Psi_2 + \frac{q^{g-1}}{1+q^{g-1}} \Psi_1$$

The elements α_r, $r \geq 1$, generate Λ_q^{nc} has a $\mathbb{C}(q)$-algebra.

⇒ abstract existence probably easy.
⇒ I would like explicit formulas.

Trivial deformation. $1_{\mathbb{Z}} \otimes \mathbb{Z}[q,q^{-1}]$
as bialgebras

(B) Commutative symmetric functions : extremely rich theory

* beautiful combinatorics
* interactions with algebraic geometry.
* rep. theory of $\mathfrak{S}_n$, $n \geq 1$ symmetric group.

avoid things of the form $x_1 + x_2^2 + x_3^3 + x_4^4$ with diverging powers

as graded algebras

$$\mathbb{C}[\underline{x}] := \mathbb{C}[x_1, x_2, \dots] := \varprojlim_n \mathbb{C}[x_1, \dots, x_n]$$

$\mathfrak{S}_\infty$ $\mathfrak{S}_n$

transition maps $\mathbb{C}[x_1, \dots, x_{n+1}] \to \mathbb{C}[x_1, \dots, x_n]$

$$x_{n+1} \mapsto 0$$

$$\Lambda = \mathbb{C}[x_1, \dots]^{\mathfrak{S}_\infty}$$

elementary symmetric functions

$$e_k = \sum_{i_1 < \dots < i_k} x_{i_1} \cdots x_{i_k}$$

power sum symmetric functions

$$p_k = \sum_i x_i^k$$

Hopf algebra :

Λ is a Hopf algebra

$$\varprojlim \mathbb{C}[x_1, \dots, x_{2n}] \to \varprojlim \mathbb{C}[x_2, x_4, \dots, x_n] \otimes \mathbb{C}[x_1, x_3, \dots, x_{2n-1}]$$

Prop: $$\Delta e_k = \sum_{u+v=k} e_u \otimes e_v$$

$$\Delta p_k = p_k \otimes 1 + 1 \otimes p_k$$

q - version.

$$\Lambda_q := \mathbb{C}(q)[x_{v}, \dots]^{\mathfrak{S}_\infty}$$

(C) New generators for $U(\mathfrak{n}_Q^+)$ [enveloping algebra]

generators of $U(\mathfrak{n}_Q^+)_i \quad i \in Q_0^{im}$.

$$\begin{array}{c} \cong \\ \left\{ \begin{array}{lll} \Lambda & g_i = 1 & \ni \Lambda_r, \; r \geq 1 \\ \Lambda^{nc} & g_i > 1 & \ni \Lambda_r \quad r \geq 1 \end{array} \right. \end{array}$$

→ better generators from a geometric point of view.

→ I think that already for quivers with one vertex, there should be a geometric understanding of Λ_q^{nc}.

⇝ $\tilde{e}_{i,n}$

Prop: The presentation of $U(\mathfrak{n}_Q^+)$ using the generators $\tilde{e}_{i,n}$ is the same as the one using $e_{i,n}$.

Geometric constructions of $\mathcal{U}^{\mathbb{Z}}(\mathfrak{n}_Q^+)$

① $K_0(\mathcal{P})$ Grothendieck group of a certain abelian category

② $\mathrm{Fun}^{sph}(\mathfrak{M}_Q, \mathbb{Z})$ constructible function on set of isoclasses of Q-reps

③ $\mathrm{Fun}^{sph}(\mathfrak{M}_{\Pi_Q}, \mathbb{Z})$ cstble function on the strictly seminilp cone of Π_Q

④ $H^{BM}_{top}(\mathfrak{M}_{\Pi_Q}, \mathbb{Z})$ top-BM homology of the ssn cone.

$\mathfrak{M}_Q$ = stack of representations of Q

$$= \bigsqcup_{d \in \mathbb{N}^{Q_0}} \mathfrak{M}_{Q,d}.$$

①-④ have natural associative algebra structures

sph means that we take the respective subalgebra generated by

$\mathbb{1}_{\mathfrak{M}_{Q, ne_i}}$; $n \geq 1$ and $i \in Q_0$.

category of perverse sheaves stable under Verdier duality.

$$\begin{tikzcd}
\mathcal{U}^{\mathbb{Z}}(\mathfrak{n}_Q^+) \arrow[r] \arrow[rr, bend left] \arrow[dr] \arrow[ddr, bend right] & K_0(\mathcal{P})^{\psi} \arrow[r, "cc"] \arrow[d, "\chi"] \arrow[dr, phantom, "\circlearrowright"] & H^{BM}_{top}(\mathfrak{M}_{\Pi_Q}, \mathbb{Z})^{\psi}_{F} \\
& \mathrm{Fun}^{sph}(\mathfrak{M}_Q, \mathbb{Z})^{\psi} \arrow[ur, "Eu^{-1} \circ T^*_{[-]}\mathfrak{M}_Q"'] & \\
& \mathrm{Fun}^{sph}(\mathfrak{M}_Q, \mathbb{Z})^{\psi} \arrow[u, "res"] \arrow[uur, dashed, bend right, "?"'] &
\end{tikzcd}$$

* Fourier transformed version
* Other perverse sheaves / constructible complexes:

$g \geq 2$ $\mathcal{P}(g)$

$\forall$ composition $\underline{n} = (n_1, \ldots, n_k)$

$S_{\underline{n}}$ = uniserial reps of g with simple subquotients given by $\underline{n}$.

$j_* \underline{\mathbb{Q}}_{S_{\underline{n}}}$

$j_! \mathbb{Q}_{S_{\underline{n}}}$

* Not clear whether these constructible complexes (or their classes) are in $K_0(\mathcal{P})$.

* projection on $K_0(\mathcal{P})$ instead?

* if $j_! \mathbb{Q}_{S_{\underline{n}}}$ is indeed in $K_0(\mathcal{P})$, it corresponds to a primitive element. when $\underline{n} = (n_1)$

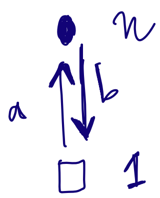
By Verdier duality $j_* \mathbb{Q}_{S_{\underline{n}}}$ is also primitive, $\underline{n} = (n_1)$.
(composition of length 1)

Appendix for myself

- Hall alg & symm fcts:
  Shimoji, Yanagida – A study of symm fcts via derived Hall algebra.

- ı Hall algebras and 2-parameters symmetric functions
  Lu, Ruan, Wang.
  derived Hall algebra Jordan quiver.

- Representations of $U(\mathfrak{g})$ using NQV
  BPS sheaf should give same thing as stable.



$ab = 0$

if n is big,

$a = 0$ $\quad \mathbb{C}^n \sqcup_{pt} \mathbb{C}^n$

if no semistables, BPS vanishes by Toda

can try to compute the kernel.

gen by $f_i^{\langle \Lambda, \alpha_i^\vee \rangle + 1}$ pairing root

highest weight

$\lambda = \Lambda_w - \alpha_{v_0}$

K-theoretical Hall algebra

$\sum w_i \Lambda_i$ $\quad \sum v_i \alpha_i$ $\quad \sum m_j \alpha_j^\vee$

$$\bigoplus_{d \in \mathbb{N}^{Q_0}} K^{G_d \times \mathbb{C}^*}(\Lambda_d)$$

$$\left\langle \sum w_i \Lambda_i - \sum v_i \alpha_i , \alpha_j^\vee \right\rangle$$

$$= \sum w_i m_i - \sum_{i,j} v_i m_j a_{ij}$$

has trivial action

$\to$ kernel

Some map of perverse sheaves vanishes

look locally