M-SEMINAR
COHOMOLOGICAL HALL ALGEBRAS OF 2-CALABI–YAU CATEGORIES AND APPLICATIONS

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Abstract. In this series of four lectures, I will explain the interactions between cohomological Hall algebras (CoHAs) and several questions of interest in algebraic geometry (in particular enumerative geometry) and representation theory (Kac-Moody algebras and their representations). CoHAs are associative algebra structures on the Borel-Moore homology of the stack of objects in some Abelian categories. We consider the CoHAs of various categories: sheaves on surfaces, representations of quivers, and representations of fundamental groups, which are 2-Calabi-Yau. CoHAs lead to a fine understanding of the cohomology of the stacks and moduli spaces involved. They provide tools to study various conjectures in the subject: cohomological integrality, positivity, and purity. In the first two lectures, I will detail how CoHAs give a geometric construction of generalised Kac-Moody algebras (in the sense of Borcherds). The last two lectures will develop applications of CoHAs to the study of the cohomology of quiver varieties (following the groundbreaking work of Nakajima from the 1990s) and to nonabelian Hodge theory (following questions of Simpson). The four lectures will, to a large extent, be independent from each other and are largely based on joint work with Ben Davison and Sebastian Schlegel Mejia.

0. Motivations

(1) Motivation: study of Nakajima quiver varieties, properties of counting polynomials associated to the constructible Hall algebra of a quiver, Nonabelian Hodge theory, enumerative geometry
(2) Borel–Moore homology: more responsive to the singularities, motivic.
(3) The two main theorems I would like to explain: the structure of the BPS associative algebra of a 2-Calabi–Yau category, the PBW isomorphism. More precisely, sheafified versions of these theorems and definition of the BPS Lie algebra for any suitable 2-Calabi–Yau Abelian category.
(4) Perspective from 3-Calabi–Yau geometry: computation of the BPS Lie algebra of any 3-CY completion of a 2CY category.

1. Constructible derived category

(1) Constructible sheaves and derived category (unbounded on the right), cohomology functors, 6-functor formalism, Verdier duality, perverse sheaves, some examples and classification of simple perverse sheaves, perverse truncations and cohomology functors, the decomposition theorem, mixed Hodge modules,
(2) monoidal structures on constructible derived categories, algebra and Lie algebra objects, free algebra, Schur functors, symmetric algebra, enveloping algebras, PBW theorem, monoidal functors.

2. Geometry of 2-Calabi–Yau categories

(1) Examples and stacks of objects: Preprojective algebras and their dg versions, sheaves on quasiprojective symplectic surfaces, fundamental group algebras of Riemann surfaces, more generally multiplicative preprojective algebras (and their derived versions)
(2) Ambient dg-category, 2-dimensional Abelian category, 2-Calabi–Yau Abelian categories: moduli stack, good moduli space, direct sum morphism, 2-Calabi–Yau structures, RHom complex and stack

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of short exact sequences, Ext-quivers, Formality of the Yoneda algebra, the local neighbourhood theorem, compatibility with the RHom-complexes.

3. $2d$ cohomological Hall algebra and BPS algebra

(1) (recalled from Lecture 1) 2-dimensional Abelian category, moduli stack and RHom complex, stack of short exact sequences, Euler form: virtual rank of the RHom complex.
(2) The constructible complexes underlying the CoHA, virtual shift, $2d$-CoHA structure: pullback, pushforward, graded multiplication, of degree 0 if Euler form is symmetric, twist of the multiplication.
(3) BPS associative algebra: truncation functor and finiteness of the direct sum.

4. A glimpse into cohomological Hall algebras of quivers with potential

(1) quiver with potential, Donaldson–Thomas sheaf, BPS Lie algebra and BPS sheaf, PBW theorem,
(2) triple quiver, canonical potential
(3) Dimensional reduction of the CoHA, support theorem for the BPS sheaf and dimensional reduction of the BPS sheaf, $2d$ PBW theorem, purity of the BPS sheaf for quivers: symmetry argument. Purity of the BPS sheaf of any 2-Calabi–Yau Abelian category.

5. Generalized Kac–Moody algebras

(1) GKM, grading monoid, roots, generators and relations, triangular decomposition, trichotomy of roots: real, imaginary and hyperbolic, examples of GKMs: Heisenberg Lie algebra, Kac–Moody algebras, lowest representations of GKMs,
(2) Positive parts of GKMs in tensor categories of perverse sheaves, punctual connected components for real roots, Serre relations, Serre ideal.

6. The BPS algebra by generators and relations

(1) Semisimplicity of the BPS sheaf, roots, generators (generating perverse sheaves): hyperbolic/real roots and isotropic roots.
(2) BPS algebra by generators and relations: Theorem A, definition of the BPS Lie algebra of any suitable 2-Calabi–Yau category
(3) Theorem B: PBW theorem.

7. The strictly seminilpotent CoHA

(1) CoHA for a saturated submonoid, interesting such monoids: nilpotent representations, strictly seminilpotent representations
(2) top CoHA of the stack of strictly seminilpotent representations: less perverse filtration, description by generators and relations, restriction of the generators to the strictly seminilpotent locus.

8. Applications

(1) Nonabelian Hodge isomorphisms for stacks
(2) Decomposition of the cohomology of Nakajima quiver varieties
(3) Positivity of cuspidal polynomials of quivers