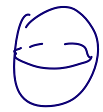
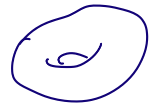
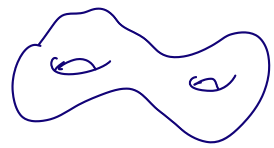
The top-CoHA of a curve

X smooth proj curve

$\mathbb{P}^1$ 

E 

C 

Coherent sheaves on X:

$$\mathcal{C}oh\,(X) = \bigsqcup_{\alpha \in \mathbb{Z}^+} \mathcal{C}oh_\alpha(X)$$

smooth, locally of finite type

$$\mathcal{C}oh_\alpha(X) = \bigcup_{\mathcal{L} \text{ line bundle}} \underline{\mathcal{C}oh_\alpha^{>\mathcal{L}}(X)}$$

coherent sheaves "strongly generated" by $\mathcal{L}$.

finite type, open substack of $\mathcal{C}oh_\alpha^{>\mathcal{L}}(X)$.

the canonical morphism $\mathrm{Hom}(\mathcal{L},\mathcal{F}) \to \mathcal{F}$ is surjective and $\mathrm{Ext}^1(\mathcal{L},\mathcal{F}) = 0$.

open subscheme of a Quot scheme

$$\mathcal{C}oh_\alpha^{>\mathcal{L}}(X) = \mathrm{Quot}(\mathcal{L},\alpha)^\circ \,/\, G_\alpha$$

$$\mathrm{Quot}(\mathcal{L},\alpha)^\circ =: Q^\circ_{\alpha,\mathcal{L}}$$

Higgs sheaves

$$\mathrm{Higgs}(X) = \bigsqcup_{\alpha \in \mathbb{Z}^+} \underbrace{\mathrm{Higgs}_\alpha(X)}_{\text{Higgs sheaves}}$$

* $\mathcal{F} \longrightarrow \mathcal{F} \otimes K_X$ $\mathcal{O}_X$-modules morphism.

* construction by symplectic reduction:

$G_\alpha \circlearrowright T^* Q^\circ_{\alpha, \mathcal{L}}$ with moment map

$$\mu_\alpha : T^* Q^\circ_{\alpha, \mathcal{L}} \longrightarrow \mathfrak{g}_\alpha$$

$$\mathrm{Higgs}_\alpha^{>\mathcal{L}}(X) := \mu_\alpha^{-1}(0) / G_\alpha$$

$$\mathrm{Higgs}_\alpha(X) = \bigcup_{\mathcal{L}\text{ l.b.}} \mathrm{Higgs}_\alpha^{>\mathcal{L}}(X)$$

$\mathcal{L}$ l. bundle.

What is Hamiltonian reduction doing (general fact)

$$\mu_\alpha^{-1}(0) = \bigcup_{\substack{O \subset Q^\circ_\alpha \\ G_\alpha\text{-orbit}}} T^*_O Q^\circ_\alpha$$

(infinite union in general)

- $\mathrm{Higgs}_\alpha(X) =$ "$T^*\mathrm{Coh}_\alpha(X)$".

  actually, 0-truncation.

  it is not (in general) equidimensional

  (these issues can be dealt with with derived geometry)

- Global nilpotent cone

$\mathcal{N} \subset \mathrm{Higgs}$ closed, Lagrangian, conical substack.

$(\mathcal{F}, \theta)$ with θ-nilpotent

in general, not reduced: irr. comps can have multiplicities (see recent work of Hausel-Hitchin)

$$\begin{tikzcd}
\mathcal{F} \arrow[r] \arrow[rrrr, bend right, "0"'] & \mathcal{F} \otimes K_X \arrow[r] & \mathcal{F} \otimes K_X^{\otimes 2} \arrow[r] & \cdots \arrow[r] & \mathcal{F} \otimes K_X^{\otimes r}
\end{tikzcd}$$

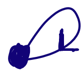
$\mathcal{N}_\alpha$ has many irreducible components which intersect in a highly nontrivial and poorly understood way, even in the simplest case of torsion sheaves.

$\mathcal{T}_d$ = torsion sheaves of degree d

$\cup$

$\mathcal{T}_{d,x}$ degree d torsion sheaves supported at x

$\cong$

nilpotent representations of 

Irreducible components of $\mathcal{N}_\alpha$

description due to Bozec, using Jordan types for Higgs sheaves.

I recall this description briefly.

- Jordan type $(\mathcal{F}, \theta)$ nilpotent Higgs sheaf

s nilpotency order of θ

$$\alpha_k = \ker \left[\mathcal{F}_{k-1}/\mathcal{F}_k \, ((k-1)K_X) \longrightarrow \mathcal{F}_k/\mathcal{F}_{k+1} \, (kK_X) \right]$$

$$\mathcal{F}_k = \operatorname{Im} \theta^k (-k\Omega)$$

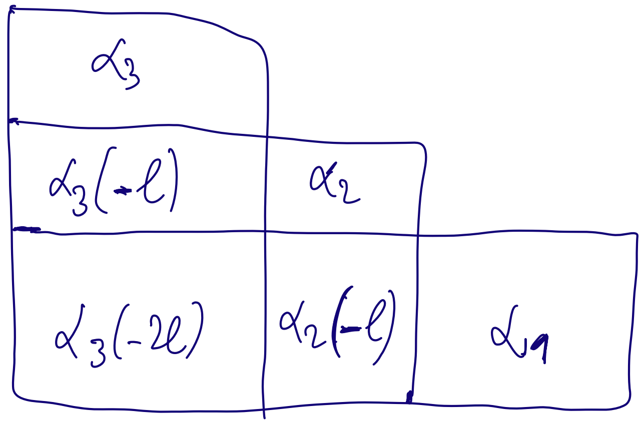
$(\alpha_1, \dots, \alpha_s)$ is the Jordan type of $(\mathcal{F}, \theta)$

Remark

If we carry out the same procedure for a nilpotent endomorphism θ of a vector space, then α_i is the number of Jordan blocks of size i of θ.

If $\alpha \in \mathbb{Z}^+$, $\ell = 2g-2 = \text{degree } K_X$, we let
$$\alpha(k\ell) = (rk(\alpha), \deg(\alpha) + rk(\alpha) \cdot k \cdot \ell).$$

If $\mathcal{F} \in \mathrm{Coh}_\alpha$, $\mathcal{F} \otimes K_X^{\otimes k}$ has class $\alpha(k\ell)$.



- $\sum$ boxes $= \alpha$
- Can read the types of successive images on this diagram.
  kernels
- semistability: the slopes of subdiagrams saturated in the directions $\leftarrow$; $\downarrow$; $\gg$ is $\leq$ than the slope of the full subdiagram.

The 2D-CoHA of a curve

The underlying vector space is

$$\mathrm{CoHA}(X) = H_*^{BM}(\mathrm{Higgs}(X))$$
$$= \bigoplus_{\alpha \in \mathbb{Z}^+} H_*^{BM}(\mathrm{Higgs}_\alpha(X)).$$

and we also have the nilpotent version:

$$\mathrm{CoHA}_{\mathcal{N}}(X) = H_*^{BM}(\mathcal{N}).$$

- The multiplication structure defined by Schiffmann - Sala use the local description of $\mathrm{Higgs}_\alpha(X)$ as hamiltonian reduction.
- This induces a multiplication on $\mathrm{CoHA}_{\mathcal{N}}(X)$.

We want to understand the "top" CoHA:

$$\mathrm{CoHA}^{top}_{\mathcal{N}}(X) := H^{BM}_{top}(\mathcal{N})$$
$$= \bigoplus_{\alpha \in \mathbb{Z}^+} \underbrace{H^{BM}_{top}(\mathcal{N}_\alpha)}$$

vector space having the set of irreducible components of $\mathcal{N}_\alpha$

as basis.

→ We have a combinatorial parametrization of a basis of $\mathrm{CoHA}_{\mathcal{N}}^{\mathrm{top}}(X)$.

Ultimate Goal

- Find generators and relations for this algebra.

Semistable CoHA(s):

$$\mathrm{CoHA}^{ss}(X) = H_*^{BM}(\mathrm{Higgs}^{ss}(X))$$

$$\mathrm{CoHA}^{ss}_{\mathcal{N}}(X) = H_*^{BM}(\Lambda^{ss})$$

Bozec characterized irreducible components of Λ^{ss} meeting the semistable locus $\mathrm{Higgs}^{ss}(X)$.

Again we are first interested in understanding the "top" semistable CoHA:

$$\mathrm{CoHA}^{ss\text{-}top}_{\mathcal{N}}(X).$$

Conjecture: ($g \geq 2$) (1) $\mathrm{CoHA}^{ss}(X)$ / $\mathrm{CoHA}^{ss}_{\mathcal{N}}(X)$ are free algebras (generated by the IC of the coarse moduli spaces)

$$\Downarrow$$

$\mathrm{CoHA}^{ss,top}_{\mathcal{N}}(X)$ is a free algebra, generated by primitive elements.

(1) has been checked by Sebastian in rk 2.

Generators

- $\mathcal{N}_\alpha \supset \mathrm{Coh}_\alpha$ as Higgs sheaves of the form $(\mathcal{F}, 0)$. It gives an irreducible components of $\mathcal{N}_\alpha$.

The classes $[\mathrm{Coh}_\alpha]$ of these irreducible components generate $\mathrm{CoHA}_{\mathcal{N}}$ as an algebra. (a topological algebra actually)

- $\mathrm{CoHA}_{\mathcal{N}}$ generated by $[\mathrm{Coh}_\alpha]$ $\mathrm{rk}(\alpha) \leq 1$?

We tried to prove this, unsuccessfully.

Problem: $\chi(\mathrm{Jac}(X)) = 0$!

$$\chi(\mathrm{Jac}(X)) = \chi\left(\Lambda^*(H^1(X,\mathbb{C}))\right)$$

. Strategy to get rid of this problem: work relatively over the Deligne–Mumford stack of genus g (≥ 2) curves.

. relative CoHA, relative characteristic cycle.

The coefficients of the characteristic cycle are not numbers (i.e. elements of $K_0(D^b(\mathrm{Vect}))$ anymore by rather elements of $K_0(D^b(\mathrm{Rep}\,\pi_1(\mathcal{M}_g)))$

$\pi_1(\mathcal{M}_g) \simeq Sp_{2g}(\mathbb{Z})$.

$$\begin{array}{ll} \mathcal{C} & \\ \big\downarrow & \text{universal curve ;} \\ M_g & \end{array} \qquad \begin{array}{ll} \mathrm{Pic}(\mathcal{C}) & \\ \big\downarrow \pi & \text{universal Picard variety} \\ M_g & \text{smooth map} \end{array}$$

$$\pi_* \underline{\mathbb{Q}}_{\mathrm{Pic}(\mathcal{C})} \cong \bigoplus_{i=0}^{2g} \underbrace{\mathcal{H}^i\left(\pi_* \underline{\mathbb{Q}}_{\mathrm{Pic}(\mathcal{C})}\right)}_{\text{local system on } M_g.}[-i]$$

has non trivial class in $K_0(D^b(\mathrm{Rep}\, Sp_{2g}(\mathbb{Z})))$.

Spherical Eisenstein perverse sheaves

Schiffmann defined a family of simple perverse sheaves on $Coh(X)$.

$$\begin{tikzcd} & \mathcal{E}xact_{\alpha,\beta} \arrow[dl, "q"'] \arrow[dr, "p"] & \\ \mathcal{C}oh_\alpha \times \mathcal{C}oh_\beta & & \mathcal{C}oh_{\alpha+\beta} \end{tikzcd}$$

$\mathcal{Q}$ = cstble complexes on $\mathcal{C}oh$ stable under $p_* q^*$, shifts, taking direct summands and the induction $p_* q^*$.

$\mathcal{P} \subset \mathcal{Q}$ full subcategory of perverse sheaves.

Eisenstein monomials

$\mathcal{Q}^{mon}$: cstble complexes on $\mathcal{C}oh$ stable under $p_* q^*$, shifts, ~~taking direct summands~~ and the induction $p_* q^*$.

$\mathcal{Q}^{mon} \subset \mathcal{Q}$
full subcategory.

Description of $\mathcal{P}$!

partially.

rank 0: torsion sheaves.

$\mathcal{T}_d$ stack of rk d torsion sheaves

U open substack

$$\mathcal{T}_d^{rss} \cong S^d X \setminus \Delta \,/\, \mathbb{G}_m^d .$$

This open substack has a $\mathfrak{S}_d$-covering

$$\begin{array}{c} X^d \setminus \Delta \\ \big\downarrow p_d \\ S^d X \setminus \Delta \end{array}$$

$p_d\, \underline{\mathbb{Q}}_{X^d \setminus \Delta}$ is a direct sum of local systems on $S^d X \setminus \Delta$.

$$= \bigoplus_{\lambda \in \mathcal{P}_d} \mathcal{L}_\lambda$$

Simple objects of $\mathcal{P}_\alpha$ are $IC(\mathcal{L}_\lambda)$, $\lambda \in \mathcal{P}_\alpha$.

- In rk 1 : also an explicit description

If $\mathcal{F} \in \mathcal{P}_\alpha$, $\quad rk(\alpha) = 1$

$\alpha = (1, d)$

$\mathcal{F}$ simple,

$$Coh_\alpha = \bigsqcup_{\ell \geq 0} \underbrace{Coh_{(\alpha-(0,\ell)),(0,\ell)}}$$

rk 1 coherent sheaves of the form

$\mathcal{L}' \oplus T$

degree $\mathcal{L}' = d - \ell$

length$(T) = \ell$.

$$Coh_{(\alpha-(0,\ell)),(0,\ell)}$$

$$\downarrow p_\ell \qquad \text{smooth morphism.}$$

$$Coh^{ss}_{(\alpha-(0,\ell))} \times Coh_{(0,\ell)}$$

semistable, since $\mathrm{rk}(\alpha)=1$, this is equivalent to being a line bundle.

$$\mathcal{F} \cong p_\ell^{*}\left(\underline{\mathbb{Q}}_{\mathcal{C}oh^{ss}_{(\alpha-(0,\ell))}} \boxtimes IC(\mathcal{L}_\lambda)\right)[\ldots]$$

relative dimension of p_ℓ.

So the simple objects of $\mathcal{P}_\alpha$ are parametrized by partitions $\lambda \in \mathcal{P}$ of any length.

$\rightarrow$ We have an infinite number of them.

- From rk 2, the task is very difficult. It is ~~still~~ still possible to say something.

The characteristic cycle map

General formalism

X smooth variety

$D^b_c(X, \mathbb{Q})$ cstble derived category of X

$\mathbb{Z}[\mathrm{Lagr}(T^*X)]$

cycles Lagrangiens coniques.

$$CC : K_0(D^b_c(X, \mathbb{Q})) \longrightarrow \mathbb{Z}[\mathrm{Lagr}(T^*X)]$$

- abelian groups homomorphism.
- If $\mathcal{L}$ is a local system on X, $CC(\mathcal{L}) = [T^*_X X]$ (normalization axiom)
- functoriality w.r.t. smooth pull-backs and proper push-forwards.

Constructions

- Using Riemann-Hilbert correspondence, it is possible to use the definition of characteristic cycle for D modules.

$\mathcal{D}_X$ sheaf of differential operators on X.

$F_\bullet$ increasing filtration by the degree of diff ops

$$gr\, \mathcal{D}_X \simeq \pi_* \mathcal{O}_{T^*X} \qquad \pi : \begin{array}{c} T^*X \\ \downarrow \\ X \end{array} \quad \text{cotangent bundle.}$$

M $\mathcal{D}_X$-module over X.
It admits a "good filtration" (compatible w/ $F_\bullet$)

$$gr\, M \text{ is a } \underset{\underset{\pi_* \mathcal{O}_{T^*X}}{\parallel}}{gr\mathcal{D}_X}\text{-module}$$

gives a $\mathcal{O}_{T^*X}$-module since π is affine.

If M is regular holonomic, $\mathrm{supp}\, gr M$ is a Lagrangian cycle in T^*X.

$$CC(M) = \sum_{\Lambda \in \mathrm{Irr}\, \mathrm{Supp}(gr M)} \mathrm{mult}_\Lambda [\Lambda].$$

- definition in terms of microlocal geometry: Kashiwara-Schapira, using directions of propagation.

Functorialities

$Y \xrightarrow{f} X$ smooth or f proper

$$\begin{tikzcd}
K_0(D^b_c(X)) \arrow[r, "CC"] \arrow[d, shift right, "f^*"'] & \mathbb{Z}[\mathrm{Lag}(T^*X)] \arrow[d, shift left, "f^*"] \\
K_0(D^b_c(Y)) \arrow[r, "CC"] \arrow[u, shift right, "f_*"'] & \mathbb{Z}[\mathrm{Lag}(T^*Y)] \arrow[u, shift left, "f_*"]
\end{tikzcd}$$

how are they defined?

Cotangent correspondence

$$\begin{tikzcd}
& Y \times_X T^*X \arrow[dl, "pr_2"'] \arrow[dr, "(df)^*"] & \\
T^*X & & T^*Y
\end{tikzcd}$$

f smooth $\Rightarrow$ (df^*) is closed immersion.

pr_2 is smooth (since base change of f)

so pull-back by pr_2 and push-forward by $(df)^*$ are well-defined.

- f proper $\Rightarrow$ pr_2 proper,
- T^*Y smooth, $Y \times_X T^*X$ is v.b of Y so is smooth too

$\Rightarrow$ $(df)^*$ is local complete intersection: we have p.b. in BM-homology.

- The maps f_* and f^* between $\mathbb{Z}[Lagr\ T^*X]$ and $\mathbb{Z}[Lagr\ T^*Y]$ are defined going back and forth through this correspondence.

The CC map gives an algebra map

$$CC : \widehat{K_0(\mathcal{Q})} \longrightarrow \widehat{\mathbb{Z}[\mathrm{Irr}\, \mathcal{N}]}$$

$$\widehat{K_0(\mathcal{Q}^{mon})}$$

not trivial!

- $CC\big|_{\widehat{K_0(\mathcal{Q}^{mon})}}$ is surjective

- Questions
  - $\widehat{K_0(\mathcal{Q})}$ has a coalgebra structure
  - $\mathbb{Z}[\mathrm{Irr}\, \mathcal{N}]$ has a coalgebra structure coming from a coproduct on the CoHA. Is CC compatible?