joint w/ Ben Davison and S. Schlegel Mejia

BPS Lie algebra action on the cohomology of Nakajima quiver varieties

Q quiver $\rightsquigarrow$ Kac-Moody algebra $\mathfrak{g}_Q$ generators and relations
representation theory : highest weight modules
→ Verma modules
→ simple quotient.

Nakajima 90's : geometric realisation of the rep. of $\mathfrak{g}_Q$ by means of "quiver varieties"
↑ noncompact Hyperkähler varieties.

$\boxed{A}$ Generalised Kac-Moody algebras

$\mathfrak{h}$ $\mathbb{Q}$-vector space , $(-,-) : \mathfrak{h} \times \mathfrak{h} \to \mathbb{Q}$ bilinear form

$\Phi^+ \subset \mathfrak{h}$ set of positive roots

Assumptions : $(h_i, h_{i'}) \in \mathbb{Z}_{\leq 0}$ $\forall$ $i \neq i' \in \Phi^+$

$(h_i, h_i) \in 2\mathbb{Z}_{\leq 1}$

$\mathcal{G} = \bigoplus_{i \in \Phi^+} \mathcal{G}_i$ $\mathbb{Q}$-vector space, "space of positive Chevalley generators".
each $\mathcal{G}_i$ is $\mathbb{Z}$-graded.

Define $\mathfrak{g}_{\mathcal{G}}$ as the Lie algebra generated by

$$\mathfrak{h} \oplus \mathcal{G} \oplus \mathcal{G}^\vee$$

with the relations

* $[h, h'] = 0 \quad \forall h, h' \in \mathfrak{h}$
* $[h, \alpha_i^\vee] = \pm (h, h_i) \alpha_i^\vee \qquad \alpha_i^\vee \in \mathcal{Y}_i^\vee$
* $[\alpha_i, \alpha_j^\vee] = \delta_{ij}\, \alpha_j^\vee(\alpha_i)\, h_i$
* $[\alpha_i^\vee, -]^{1-(h_i, h_j)} (\alpha_j^\vee) = 0$ if $(h_i, h_j) = 0$ or $(h_i, h_i) = 2$.

Serre relations

Trichotomy of roots

$h_i, i \in \Phi^+$ come in 3 kinds

- real: $(h_i, h_i) = 2$
- isotropic: $(h_i, h_i) = 0$
- hyperbolic: $(h_i, h_i) < 0$

] imaginary.

Many facts true for semisimple Lie algebras remain true for GKM algebras.

- triangular decomposition $\mathfrak{g}_{\mathcal{Y}} = \mathfrak{n}_{\mathcal{Y}}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}_{\mathcal{Y}}^-$, with $\mathfrak{n}_{\mathcal{Y}}^+ = \langle \mathcal{Y} \rangle$, $\mathfrak{n}_{\mathcal{Y}}^- = \langle \mathcal{Y}^\vee \rangle$.
- $\mathfrak{n}_{\mathcal{Y}}^+$ is gen. by $\mathcal{Y}$ w/ Serre relations only.

Goal of the talk: * explaining how such Lie algebras arise from geometry.

* These Lie algebras act on the cohomology of highly relevant moduli spaces.

Examples of GKM . $Q = (Q_0, Q_1)$ quiver

$\mathfrak{h} = \mathbb{Q}^{Q_0}$

Q_0 vertices

Q_1 arrows : $s, t : Q_1 \to Q_0$ source and target

$$s(\alpha) \xrightarrow{\alpha} t(\alpha)$$

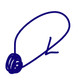
$$(d, e) = \chi_Q(d, e) + \chi_Q(e, d)$$
$$= 2 \sum_{i \in Q_0} d_i e_i - \sum_{\alpha \in Q_1} \left(d_{s(\alpha)} e_{t(\alpha)} + e_{t(\alpha)} d_{s(\alpha)} \right).$$

symmetrised Euler form of Q

① Q has no loops
$\Phi_+ = Q_0$
$\mathcal{G}_i = \mathbb{Q} \quad i \in Q_0.$
] $\mathfrak{g}_{\mathcal{G}} =: \mathfrak{g}_Q$ is the Kac-Moody algebra classically associated with Q

<u>subexample</u> : Q is Dynkin ADE : $\mathfrak{g}_{\mathcal{G}}$ is the semisimple Lie algebra associated to Q

A $\mathfrak{sl}_n$
D $\mathfrak{so}(2n)$
E exceptional

② $Q =$ 

$\Phi_+ = \mathbb{Z}_{\geq 1} \subset \mathbb{Q} =: \mathfrak{h}$
$\mathcal{G}_n = \mathbb{Q}[2] \quad \forall n \geq 1$
] $\mathfrak{g}_{\mathcal{G}} =$ heis , Heisenberg Lie algebra acting on the cohomology of Hilbert schemes of points on $\mathbb{C}^2$, infinite dimensional.

$\boxed{B}$ Cohomological Hall algebras and BPS algebras

2CY Abelian categories $\mathcal{A}$

Keep an example in mind.

<u>geometric</u>: $*$ C smooth projective curve, $\mu \in \mathbb{Q}$ slope

$\mathcal{A}$ = semistable Higgs bundles of slope μ on C

$(\mathcal{F}, \theta)$ $\quad$ $\mathcal{F}$ coh. sheaf on C

$\theta : \mathcal{F} \to \mathcal{F} \otimes \Omega^1_C$ $\quad$ Higgs field.

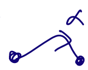
$*$ $\mathcal{A} = \mathrm{Coh}^{H\text{-sst}}_{p}(S)$ $\quad$ S symplectic surface

H polarisation

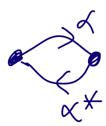
p reduced Hilbert polynomial.

<u>algebraic</u>: $\mathcal{A} = \mathrm{Rep}\, \Pi_Q$ $\quad$ rep. of the preprojective algebra of Q

Q quiver



$\overline{Q}$ double



$\mathbb{C}\overline{Q}$ = path algebra of $\overline{Q}$

$\cup$

$\rho = \sum_{\alpha \in Q_1} [\alpha, \alpha^*]$

$\Pi_Q = \mathbb{C}\overline{Q} / \langle\langle \rho \rangle\rangle$

<u>Topological</u>: $\mathcal{A} = \operatorname{Rep} \pi_1(C, x)$ $\quad x \in C$.

C genus $g \geq 2$ Riemann surface.

In $g^{\underline{al}}$: $\mathcal{A}$ such that $\operatorname{Ext}^{2-i}(M,N) \simeq \operatorname{Ext}^{i}(N,M)^*$ functorially.

<u>CoHA</u>: $\quad \mathfrak{M}_{\mathcal{A}} \xrightarrow{JH} \mathcal{M}_{\mathcal{A}}$ stack of objects and good moduli space

$\mathcal{D}^+_c(\mathcal{M}_{\mathcal{A}})$ constructible derived category of $\mathcal{M}_{\mathcal{A}}$

monoidale structure $\oplus : \mathcal{M}_{\mathcal{A}} \times \mathcal{M}_{\mathcal{A}} \longrightarrow \mathcal{M}_{\mathcal{A}}$

$$\mathcal{F} \boxdot \mathcal{G} := \oplus_* (\mathcal{F} \boxtimes \mathcal{G}).$$

$$0 \to M \to E \to N \to 0$$

$$\begin{tikzcd}
 & \mathcal{E}xact_{\mathcal{A}} \arrow[dl, "q"'] \arrow[dr, "p"] & \\
\mathfrak{M}_{\mathcal{A}} \times \mathfrak{M}_{\mathcal{A}} \arrow[d, "JH \times JH"'] \arrow[drr, phantom, "\circlearrowright"] & & \mathfrak{M}_{\mathcal{A}} \arrow[d, "JH"] \\
\mathcal{M}_{\mathcal{A}} \times \mathcal{M}_{\mathcal{A}} \arrow[rr, "\oplus"'] & & \mathcal{M}_{\mathcal{A}}
\end{tikzcd}$$

(M,N) $\qquad\qquad$ E.

$\mathcal{A} := JH_* \mathbb{D}\mathbb{Q}^{vir}_{\mathfrak{M}_{\mathcal{A}}} \in \mathcal{D}^+_c(\mathcal{M}_{\mathcal{A}})$ $\quad$ underlying cstble complex of the sheaf theoretic CoHA.

$\rightsquigarrow \quad m = \text{"}p_* q^! \text{"} \in \mathrm{Hom}_{\mathcal{D}^+_c(\mathcal{M}_{\mathcal{A}})}(\mathcal{A} \boxtimes \mathcal{A}, \mathcal{A})$.

associative algebra structure in a rich monoidal category:

$$\mathcal{A} = \left(\quad \cdots \to \mathcal{A}^{-1} \xrightarrow{d^{-1}} \mathcal{A}^0 \xrightarrow{d^0} \mathcal{A}^1 \xrightarrow{d^1} \mathcal{A}^2 \quad \right)$$

We cut a subalgebra. Cutting objects in triangulated categories is done using t-structures.

- standard t-structure
- perverse t-structure : best suited for us.

Define $\mathcal{BPS}_{\mathcal{A}} := {}^p\mathcal{H}^0(\mathcal{A})$ "sheaf-th. BPS associative algebra"

Prop : m gives a multiplication on $\mathcal{BPS}_{\mathcal{A}}$

Proof: ${}^p\mathcal{H}^i(\mathcal{A}) = 0 \quad \forall i < 0$.

<u>Going back to actual algebras</u>:

$$\pi : \mathcal{M}_{\mathcal{A}} \longrightarrow pt$$

$$\pi_* \mathcal{A} = H^{BM}_*(\mathfrak{M}_{\mathcal{A}})$$

$\cup$ subalgebra

$$\pi_* \mathcal{BPS}_{\mathcal{A}} =: \mathrm{BPS}_{\mathcal{A}}$$

$\boxed{C}$ The BPS algebra and the BPS Lie algebra

Thm (DHS) $\mathrm{BPS}_{\mathcal{A}} \simeq U(\mathfrak{n}^+_{\mathcal{A}})$ où

$\mathfrak{g}_{\mathcal{A}} = \mathfrak{n}^+_{\mathcal{A}} \oplus \mathfrak{h} \oplus \mathfrak{n}^-_{\mathcal{A}}$ is a GKM.

More precisely: $\mathfrak{h} = (\pi_0(\mathcal{M}_{\mathcal{A}}), \oplus) \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$

$(-,-) : \mathfrak{h} \otimes \mathfrak{h} \longrightarrow \mathbb{Q}$ induced by

the Euler form $\chi_{\mathcal{A}}(M,N) = \sum_{i \in \mathbb{Z}} (-1)^i \operatorname{ext}^i(M,N)$

$$\Phi^+ = \underbrace{\Sigma}_{\substack{\text{connected components}\\ \text{containing an open of simple objects}}} \cup \underbrace{I}_{\{la \,:\, l \in \mathbb{Z}_{\geq 1},\ a \in \Sigma,\ (a,a) = 0\}} \subset \pi_0(\mathcal{M}_{\mathcal{A}})$$

Generators: $\mathfrak{g}_a = \begin{cases} \mathrm{IH}(\mathcal{M}_{\mathcal{A},a}) & \text{if } a \in \Sigma \\ \mathrm{IH}(\mathcal{M}_{\mathcal{A},a'}) & \text{if } a = la',\ a' \in \Sigma,\ l \in \mathbb{Z}_{\geq 1}. \end{cases}$

Examples: C smooth proj curve genus ≥ 2

$\mathcal{A} = \mathrm{Higgs}^{0\text{-sst}}(C)$

$\pi_0(\mathcal{M}_{\mathcal{A}}) = \mathbb{Z}_{\geq 0}$ $\qquad \mathfrak{h} = \mathbb{Q}$

$(-,-) : \mathbb{Q} \times \mathbb{Q} \longrightarrow \mathbb{Q}$

$\qquad r, s \longmapsto 2(1-g)rs$

"totally negative": $(\mathbb{Q}_{>0}, \mathbb{Q}_{>0}) \subset \mathbb{Q}_{<0}$

$$\Rightarrow \quad BPS_{\mathcal{A}} \cong \mathrm{Free}_{Alg}\left(\bigoplus_{r\geq 1} IH(\mathcal{M}^{Dol}_{(r,0)}(C)) \right)$$

The BPS Lie algebra

$$BPS_{\mathcal{A},Lie} := n^+_{\mathcal{A}}$$

Since $n^+_{\mathcal{A}} \subset g_{\mathcal{A}}$ GKM, it is natural to see $g_{\mathcal{A}}$ as the "full" BPS Lie algebra.

It acts on the cohomology of the spaces of framed objects in $\mathcal{A}$ $\rightsquigarrow$ decomposition into highest weight simple modules.

e.g: in the example above, $BPS_{\mathcal{A},Lie} = \mathrm{Free}_{Lie}(IH(\mathcal{M}^{Dol}))$.

The full CoHA

Thm (DHS) We have an isomorphism of vector spaces

$$H^{BM}(\mathfrak{M}_{\mathcal{A}}) \simeq \underset{\text{total symmetric power}}{\mathrm{Sym}}\left(n^+_{\mathcal{A}} \otimes H^*(B\mathbb{C}^*) \right)$$

$\rightarrow$ gives a connection between the size of $n^+_{\mathcal{A}}$ and that of $H^{BM}(\mathfrak{M}_{\mathcal{A}})$.

Ingredients of the proofs

ⓐ Decomposition thm for 2CY categories: (Davison)

$$JH_* \mathbb{D}\mathbb{Q}^{vir}_{\mathfrak{M}} \in \mathcal{D}^+_c(\mathcal{M}_{\mathcal{A}}) \quad \text{is a semisimple complex}$$

ⓑ Neighbourhood theorem for 2CY categories (Davison)

ⓒ Description of the top-CoHA of the strictly seminilpotent cone for preprojective algebras of quiver (H)

ⓑ details

$\mathcal{A}$ 2CY $\quad \mathfrak{M}_{\mathcal{A}} \xrightarrow{JH} \mathcal{M}_{\mathcal{A}} \ni x$

x corresponds to $\mathcal{F} = \bigoplus_{i=1}^{r} \mathcal{F}_i^{m_i}$.

$\underline{\mathcal{F}} = \{\mathcal{F}_i, 1 \leq i \leq r\}$ collection of simple of $\mathcal{A}$

$\overline{Q}_{\underline{\mathcal{F}}} = (\underline{\mathcal{F}}, \text{arrows})$

$\#\{\mathcal{F}_i \to \mathcal{F}_j\} = \mathrm{ext}^1(\mathcal{F}_i, \mathcal{F}_j)$.

$\overline{Q}_{\underline{\mathcal{F}}}$ is the double of some (non-unique) quiver $Q_{\underline{\mathcal{F}}}$.

$$\begin{tikzcd}
(\mathfrak{M}_{\mathcal{A}}, x) \arrow[d, "JH"'] & (\mathcal{X}, y) \arrow[l] \arrow[r] \arrow[d] \arrow[dl, phantom, "\top"] \arrow[dr, phantom, "\ulcorner"] & (\mathfrak{M}_{\Pi_Q}, 0_m) \arrow[d, "JH"] \\
(\mathcal{M}_{\mathcal{A}}, x) & (\mathcal{U}, y) \arrow[l] \arrow[r] & (\mathcal{M}_{\Pi_Q}, 0_m)
\end{tikzcd}$$

with étale horizontal maps.

(c) details

$Q = (Q_0, Q_1)$

Lagrangian substack.

$$\begin{tikzcd}
\mathfrak{M}^{SSN}_{\Pi_Q} \arrow[r, "i"] \arrow[d, "JH^{SSN}"'] \arrow[dr, phantom, "\ulcorner", very near start] & \mathfrak{M}_{\Pi_Q} \arrow[d, "JH"] \\
\mathcal{M}^{SSN}_{\Pi_Q} \arrow[r, hook, "i"] & \mathcal{M}_{\Pi_Q}
\end{tikzcd}$$

submonoid

$$\mathcal{M}^{SSN}_{\Pi_Q} = \left\{ (x_\alpha, x_{\alpha^*}) \in \mathcal{M}_{\Pi_Q} \;\middle|\; \begin{array}{l} x_\alpha = 0 \text{ and } x_{\alpha^*} = 0 \text{ if} \\ \alpha^* \text{ is not a loop} \end{array} \right\}$$

$$H^*(i^! \mathcal{A}_{\Pi_Q}) =: H^*(\mathcal{A}^{SSN}_{\Pi_Q}) \supset H^0(\mathcal{A}^{SSN}_{\Pi_Q})$$

subalgebra.

$H^0(\mathcal{A}^{SSN}_{\Pi_Q})$ has a $\mathbb{C}$-linear basis given by irreducible components of $\mathfrak{M}^{SSN}_{\Pi_Q}$. [Bozec]

$$\mathbb{C}[\![t]\!]\, H^0(\mathcal{A}^{SSN}_{\Pi_Q}) \cong U(\tilde{\mathfrak{n}}^+_Q) \quad \text{where } \tilde{\mathfrak{n}}^+_Q \text{ is}$$

Bozec's Lie algebra of Q

[when Q has no loops, this is the KM algebra of Q].

if loops, need to take them into account.

With our formalism for GKMs,

$\mathfrak{h} = \mathbb{Q}^{Q_0}$ $\quad (-,-) : \mathfrak{h} \times \mathfrak{h} \longrightarrow \mathbb{Q}$ symmetrised Euler form of Q

$\Phi_+ = (\mathbb{Z}_{\geq} \times Q_0^{im}) \sqcup Q_0^{real}$

$Q_0^{im} \subset Q_0$
vertices with at least 1 loop
Q_0^{real} without loops

$\mathfrak{g}_i = \mathbb{Q} \quad \forall i \in \Phi_+$

$\tilde{\mathfrak{n}}_Q^+$ is the positive part of $\mathfrak{g}_{\mathcal{G}}$, GKM generated by $\mathcal{G}$.

<u>Representations of GKMs</u> (highest weight)

$\lambda \in \mathfrak{h}^*$ linear form
$\rightsquigarrow \lambda : U(\mathfrak{h}) \longrightarrow \mathbb{Q}$ 1-dimensional representation.

$$U(\mathfrak{g}_{\mathcal{G}}) \underset{\text{v. space}}{\simeq} U(\mathfrak{n}_{\mathcal{G}}^+) \otimes \underbrace{U(\mathfrak{h}) \otimes U(\mathfrak{n}_{\mathcal{G}}^-)}_{\substack{\text{subalgebra} \\ =: U(\mathfrak{b}_{\mathcal{G}}^-)}}$$

$\lambda : U(\mathfrak{b}_{\mathcal{G}}^-) \longrightarrow \mathbb{C}$ 1 dimensional rep.

$M(\lambda) := U(\mathfrak{g}_{\mathcal{G}}) \otimes_{U(\mathfrak{b}_{\mathcal{G}}^-)} \mathbb{C}$ Verma module

$L(\lambda) :=$ smallest nonzero quotient of $M(\lambda)$
exists by general considerations.