Nonabelian Hodge isomorphisms for cohomological Hall algebras

in part joint work with Davison and Schlegel Mejia

[Hitchin, Nitsure, Simpson, ...]

NAHT : connections between 3 types of geometric/top objects on a smooth projective manifold X ; later: $X = C$ smooth proj curve.
genus ≥ 2 unless otherwise specified

① Higgs bundles $\mathcal{F}$ v. bundle on X, $\theta : \mathcal{F} \longrightarrow \mathcal{F} \otimes \Omega^1_X$ $\mathcal{O}_X$-linear

Dolbeault

vanishing Chern classes.

$\theta(fs) = f\,\theta(s)$ $\forall f$ function, s section of $\mathcal{F}$,

$\theta \wedge \theta = 0$

semistable : $\dfrac{\deg(\mathcal{G})}{rk(\mathcal{G})} \leq \dfrac{\deg(\mathcal{F})}{rk(\mathcal{F})}$ $(\forall 0 \neq \mathcal{G} \subset \mathcal{F})$ for curves.

② Local systems = representations of $\pi_1(X, x)$

Betti

③ Connections = vector bundles with flat connection.

de Rham

$(\mathcal{F}, \nabla)$ $\mathcal{F}$ v. bundle, $\nabla : \mathcal{F} \longrightarrow \mathcal{F} \otimes \Omega^1_X$ satisfying the Leibniz rule

$\nabla(fs) = f\nabla s + s \otimes df$ $\forall f$ function, s section of $\mathcal{F}$.

flat : $\nabla \wedge \nabla = 0$

Simpson 1992 : bijections
Corlette
Hitchin
⋮

$$\begin{tikzcd}[column sep=small] & \left\{\begin{array}{c}\text{semistable Higgs}\\ \text{bundle w/ vanishing Chern}\\ \text{classes}\end{array}\right\}/\sim \ \text{Dolbeault} \arrow[dl, leftrightarrow, "\text{NAHT}"'] \arrow[dr, leftrightarrow, dashed] & \\ \underset{\text{de Rham}}{\left\{\begin{array}{c}\text{v.b with flat}\\ \text{connections}\end{array}\right\}/\sim} \arrow[rr, leftrightarrow, "\text{Riemann-Hilbert}"'] & & \underset{\text{Betti}}{\left\{\begin{array}{c}\text{representations of}\\ \pi_1(X,x)\end{array}\right\}/\sim} \end{tikzcd}$$

Each side has moduli. NAHT (also) gives comparisons between this moduli spaces.

Construction : * consider framed objects
* take the quotient by the group changing the framing.

<u>e.g.</u> ② Betti side. $\mathcal{L}$ local system of rank r on $X \ni x$ fixed
framing is $\mathcal{L}_x \simeq \mathbb{C}^r$.

A framed local system is $\pi_1(X,x) \longrightarrow GL_r(\mathbb{C})$

$$\rightsquigarrow \operatorname{Hom}_{grp}(\pi_1(X,x), GL_r(\mathbb{C})) \quad \text{finitely presented group}$$

$$\overset{\text{finitely presented (affine) } \mathbb{C}\text{-scheme}}{=} \qquad = R_r^B$$

$X = C \qquad \pi_1(X,x) = \left\langle \underset{1\leq i\leq g}{x_i, y_i} \;\middle|\; \prod_{i=1}^{g} x_i y_i x_i^{-1} y_i^{-1} = 1 \right\rangle.$

Action of $Gl_r(\mathbb{C})$ by conjugation

Betti stack: $\mathfrak{M}_r^B = \mathrm{Hom}(\pi_1(X,x), Gl_r(\mathbb{C}))/Gl_r(\mathbb{C})$.

Betti moduli space = character variety:

$$M_r^B = \mathrm{Hom}(\pi_1(X,x), Gl_r(\mathbb{C}))/\!/Gl_r(\mathbb{C}).$$

$JH : \mathfrak{M}_r^B \longrightarrow M_r^B$ Jordan–Hölder map / good moduli space.

(1) Dolbeault: $JH: \mathfrak{M}_{r,0}^{Dol}(X) \longrightarrow M_{r,0}^{Dol}(X)$

$$\mathfrak{M}_{r,0}^{Dol}(X) = R_r^{Dol}(X)/Gl_r, \qquad M_{r,0}^{Dol}(X) = R_r^{Dol}/\!/Gl_r.$$

$$R_r^{Dol}(X) = \{\mathcal{F}, \theta, \beta : \mathcal{F}_x \simeq \mathbb{C}^r\}$$

(3) de Rham: $JH: \mathfrak{M}_r^{dR}(X) \longrightarrow M_r^{dR}(X)$.

$$\mathfrak{M}_r^{dR}(X) = R_r^{dR}/Gl_r, \qquad M_r^{dR}(X) = R_r^{dR}/\!/Gl_r$$

$$R_r^{dR} = \{\mathcal{F}, \nabla, \beta : \mathcal{F}_x \simeq \mathbb{C}^r\}$$

$R_r^{Dol}(X)$, R_r^{dR} are actual schemes, parametrising framed objects.

NAHT picture [moduli space version of the previous diagram]

$$\begin{tikzcd}
 & & R_r^{Dol} \arrow[ddll, leftrightarrow, "1\ \ 1:1\ \text{set-theoretic}"'] \arrow[ddrr, leftrightarrow, "2\ \ 1:1,\ \text{set-theoretic bijection between } \mathbb{C}\text{-points}"] \arrow[d] & & \\
 & & M_r^{Dol} \arrow[dl, leftrightarrow, "\sim"'] \arrow[dr, leftrightarrow, "\sim"] & & \\
R_r^{dR} \arrow[r] \arrow[rrrr, leftrightarrow, bend right, "RH\ \ 1:1"'] & M_r^{dR} \arrow[rr, leftrightarrow, "\sim"'] & & M_r^B & R_r^B \arrow[l]
\end{tikzcd}$$

r. an isos

still: homeomorphism between stable loci.

real-analytic isomorphism, Gl_r-equivariant

Simpson: The maps ①, ② are not continuous, explicit counterexample.
Simpson's c.ex is for $X = E$ elliptic curve.

Inner triangle: we (obviously) have isos

$$H^*(\mathcal{M}_r^{Dol}) \cong H^*(\mathcal{M}_r^{B}) \cong H^*(\mathcal{M}_r^{dR})$$

$$H_*^{BM}(\mathcal{M}_r^{Dol}) \cong H_*^{BM}(\mathcal{M}_r^{B}) \cong H_*^{BM}(\mathcal{M}_r^{dR}).$$

Also, RH induces isos $H^*_{GL_r}(R_r^{dR}) \cong H^*_{GL_r}(R_r^{B})$

$$H_*^{BM, GL_r}(R_r^{dR}) \cong H_*^{BM, GL_r}(R_r^{B})$$

Z space $H^*(Z) := H^*(X, \mathbb{Q}_Z)$
$H_*^{BM} = H^{-*}(Z, \mathbb{D}\mathbb{Q}_Z)$
dualising complex

Question: Is it possible to compare H^* / H_*^{BM} for $\mathfrak{M}_r^{Dol}$ and $\mathfrak{M}_r^{B} / \mathfrak{M}_r^{dR}$?

Answer: Hard question in general!
Take X a smooth projective curve
Yes: D, Davison, H, Schlegel Mejia, for H_*^{BM}.

X = Curve is already a highly non trivial situation.
$\rightsquigarrow$ $P=W$ conjecture relates filtrations on

$$H^*(\mathcal{M}_{r,d}^{Dol}(C)) \cong H^*(\mathcal{M}_{r,d}^{B}(C)) \quad [(r,d)=1]$$

New question: $H_*^{BM}(\mathfrak{M}_r^{\#}(C))$ has more structure: algebra structure coming from the CoHA construction.
Can we compare the algebra structures?

Answer: Yes!

Thm: The CoHAs $\bigoplus_{r\geq 0} H_*^{BM}(\mathfrak{M}_r^{\#}(C))$ for $\# \in \{Dol, B, dR\}$ are all isomorphic.

Example: • $C = \mathbb{P}^1$: $\mathbb{P}^1 \simeq S^2 \simeq \mathbb{R}^2 \cup \{\infty\}$ is simply connected and so for any $r \in \mathbb{N}$, there is a unique (up to iso) local system / flat connection of rank r.

• Also, the unique semistable Higgs bundle of degree 0 is and rank r is $\mathcal{O}_C^{\oplus r}$.

In any case, $\mathfrak{M}_r^{\#}(C) \simeq pt/GL_r$ $\# \{Dol, B, dR\}$.

$\to$ comparison of homologies/cohomologies is obvious.

What about the algebra structures?

Roughly, we want to compare the derived structures on $\mathfrak{M}_r^{\#}(C)$.

Fact: It almost suffices to compare the tangent complexes to the derived stacks.

$\mathbb{T}_{pt}\mathfrak{M}_r^{\#}(C) \simeq \mathfrak{gl}_r \to 0 \to \mathfrak{gl}_r$

$\Rightarrow$ the algebra structures coincide.

by generators and relations

Going further: Describe explicitly the algebra $\bigoplus_{r\geq 0} H_*^{BM}(\mathfrak{M}_r^{\#}(C))$.

That's a hard question!

$\boxed{g=0} \rightsquigarrow U(\mathbb{C}[x]) \simeq Sym(\mathbb{C}[x])$

$\simeq \bigoplus_{r\geq 0} \mathbb{C}[x_1, \dots, x_r]$ and product is given by $\mathbb{C}[x_1, \dots, x_r] \otimes \mathbb{C}[x_1, \dots, x_s] \longrightarrow \mathbb{C}[x_1, \dots, x_{r+s}]$.

$\boxed{g=1}$ Already trickier. Fourier-Mukai transform is the key.

It provides isos $\mathfrak{M}^{Dol} \simeq \mathfrak{M}_{torsion}$

$\uparrow$ stack of $(\mathcal{F}, \theta)$, $\mathcal{F}$ torsion, $\theta \in End(\mathcal{F})$.

CoHA for torsion Higgs sheaves $=$ $CoHA_{f.l}(T^*C)$ is know by work of [MMSV] in progress

related to $W_{1+\infty}$ Lie algebra.

see also [Davison] for $CoHA_{f.l}(\mathbb{A}^2)$. $\mathbb{A}^2 \cong T^*\mathbb{A}^1$

CoHA structures [idea: Kontsevich-Soibelman, Schiffmann-Vasserot]

$\mathcal{A}$ Abelian category with stack of objects $\mathfrak{M}_{\mathcal{A}}$.
stack of short exact sequences $\mathcal{E}xact_{\mathcal{A}}$

$$\begin{tikzcd} & \mathcal{E}xact_{\mathcal{A}} \arrow[dl, "q"'] \arrow[dr, "p"] & \\ \mathfrak{M}_{\mathcal{A}} \times \mathfrak{M}_{\mathcal{A}} & & \mathfrak{M}_{\mathcal{A}} \end{tikzcd}$$

$$0 \to N \to E \to M \to 0$$
$$(M, N) \qquad\qquad E$$

* p proper
* q regular enough*

Then, $m = p_* q^* : H_*^{BM}(\mathfrak{M}_{\mathcal{A}}) \otimes H_*^{BM}(\mathfrak{M}_{\mathcal{A}}) \longrightarrow H_*^{BM}(\mathfrak{M}_{\mathcal{A}})$ gives an associative algebra structure.

* q regular enough is the condition to be able to construct the pullback map $\mathbb{D}\mathbb{Q}_{\mathfrak{M}_{\mathcal{A}} \times \mathfrak{M}_{\mathcal{A}}} \longrightarrow q_* \mathbb{D}\mathbb{Q}_{\mathcal{E}xact_{\mathcal{A}}}[\text{shift}]$

q "quasi-smooth": q comes from a map between derived stacks
$q : \underline{\mathcal{E}xact}_{\mathcal{A}} \to \underline{\mathfrak{M}}_{\mathcal{A}} \times \underline{\mathfrak{M}}_{\mathcal{A}}$ s.t. $\mathbb{T}_q$ is perfect in degrees ≤ 1
in this case, shift $= \pm$ virtual rank of $\mathbb{T}_q$
virtual relative dimension of q.

for us: homological dimension of $\mathcal{A}$ is
0
$1 \Longrightarrow H_*^{BM}(\mathfrak{M}_{\mathcal{A}})$ is a shuffle algebra
$2 \longrightarrow H_*^{BM}(\mathfrak{M}_{\mathcal{A}})$ is a "Yangian"
i.e. $\cong U(\underbrace{\mathfrak{g}[u]}_{})$ after taking associated graded
Lie alg of pols w/ coeffs in $\mathfrak{g}$.

Sheafified/Relative/enriched CoHA and BPS algebras

Possible to work over $\mathcal{M}_{\mathcal{A}}$

good moduli space
Alper-Hall-Rydh
behaves like a GIT quotient

$$\begin{tikzcd} \mathfrak{M}_{\mathcal{A}} \arrow[r, "JH"] \arrow[rr, bend right] & \mathcal{M}_{\mathcal{A}} \arrow[r, "p"] & pt \end{tikzcd}$$

$$\mathcal{A} := JH_* \mathbb{D}\mathbb{Q}^{vir}_{\mathfrak{M}_{\mathcal{A}}} \in \mathcal{D}_c^+(\mathcal{M}_{\mathcal{A}})$$
$$p_* \mathcal{A} = H_*^{BM}(\mathfrak{M}_{\mathcal{A}})$$

Use the diagram:

$$\begin{tikzcd}
 & \mathfrak{Exact}_{\mathcal{A}} \arrow[dl, "q"'] \arrow[dr, "p"] & \\
\mathfrak{M}_{\mathcal{A}} \times \mathfrak{M}_{\mathcal{A}} \arrow[d, "JH \times JH"'] \arrow[rr, phantom, "\circlearrowleft"] & & \mathfrak{M}_{\mathcal{A}} \arrow[d, "JH"] \\
\mathcal{M}_{\mathcal{A}} \times \mathcal{M}_{\mathcal{A}} \arrow[rr, "\oplus"'] & & \mathcal{M}_{\mathcal{A}}
\end{tikzcd}$$

(Davison) $\mathcal{A}$ 2CY Abelian category.

BPS algebra

Prop $\mathcal{A}$ is concentrated in nonnegative perverse degrees: ${}^p\mathcal{H}^i(\mathcal{A}) = 0$ if $i<0$

$\Rightarrow$ ${}^p\mathcal{H}^0(\mathcal{A}) =: \mathcal{BPS}_{\mathcal{A},\mathrm{Alg}} \in \mathrm{Perv}(\mathcal{M}_{\mathcal{A}})$ is an associative algebra object.

Crucial in the comparison of CoHAs for NAHT

Theorem (DHS) Let $\mathcal{A}$ be a totally negative 2CY Abelian category.

- $\mathfrak{M}_{\mathcal{A}}$ moduli stack of objects
- $JH : \mathfrak{M}_{\mathcal{A}} \longrightarrow \mathcal{M}_{\mathcal{A}}$ good moduli space
- $\mathrm{Ext}^{2-i}(M,N) \simeq \mathrm{Ext}^i(N,M)^*$ functorially (2CY)
- $\chi(M,N) = \sum_{i\in\mathbb{Z}} (-1)^i \mathrm{ext}^i(M,N) < 0$ if $M,N \neq 0$ (tot. negative)

$\Sigma_{\mathcal{A}} \subset \pi_0(\mathfrak{M}_{\mathcal{A}})$ set of conn. components containing a simple rep of $\mathcal{A}$.

Then: $\mathcal{BPS}_{\mathcal{A},\mathrm{Alg}} \cong \underset{\text{env. alg}}{\mathbf{U}}\Big(\underset{\substack{\text{Lie alg object}\\ \text{in } \mathrm{Perv}(\mathcal{M}_{\mathcal{A}})}}{\mathcal{BPS}_{\mathcal{A},\mathrm{Lie}}}\Big)$

where $\mathcal{BPS}_{\mathcal{A},\mathrm{Lie}} := \mathrm{FreeLie}_{\boxdot}\left(\bigoplus_{a\in\Sigma_{\mathcal{A}}} \mathcal{IC}(\mathcal{M}_{\mathcal{A},a})\right)$.

Free Lie algebra.

and (PBW-type iso)

$$\mathcal{A}_{\mathcal{A}} \cong \mathrm{Sym}_{\boxdot}\left(\mathcal{BPS}_{\mathcal{A},\mathrm{Lie}} \otimes H^*_{\mathbb{C}^*}\right)$$

as constructible complexes.

Proof: hard, not the subject of this talk.

examples:
* $\mathrm{Rep}\,\pi_1(C, x)$ $\quad g(C) \geq 2$
* $\mathrm{Conn}(C)$
* $\mathrm{Higgs}^{ss}_{\mu}(C)$ $\quad \mu \in \mathbb{Q}$ slope

CoHAs defined by Davison, Porta-Sala, Sala-Schiffmann in the non-sheafified setting.

* Euler form: $(\mathcal{F}, \mathcal{G}) = 2(1-g)\,\mathrm{rk}(\mathcal{F})\,\mathrm{rk}(\mathcal{G})$ in all three cases. ($\Rightarrow$ totally negative)
* $\pi_0(\mathfrak{M}^{\#}) \simeq \mathbb{N}$
* $\Sigma_{\mathcal{A}} = \mathbb{N}\setminus\{0\}$ in all 3 cases since we have simples objects in all ranks ($g(C) \geq 2$).

Recall: <u>Theorem</u>: $H_*^{BM}(\mathfrak{M}^\#)$ $\# \in \{Dol, Betti, dR\}$ have algebra structures and they are all isomorphic through NAHT.

$\boxed{\text{Idea of proof}}$: * The derived Riemann-Hilbert correspondence gives an isomorphism of derived analytic stacks

$$\mathfrak{M}_r^{dR}(C) \simeq \mathfrak{M}_r^{B}(C)$$

and the derived structure knows how the CoHA multiplication is constructed: this is enough.

* Hodge moduli stack $\mathfrak{M}_r^{Hod}(C)$ parametrising λ-connections

$$\begin{array}{c}\mathfrak{M}_r^{Hod}(C)\\ \downarrow \\ \mathbb{A}^1\end{array}$$

w/ varying λ.

<u>Recall</u>: a λ-connection is $(\mathcal{F}, \nabla)$ where $\mathcal{F}$ is a v.b on C and $\nabla : \mathcal{F} \longrightarrow \mathcal{F} \otimes \Omega_C^1$ satisfies the λ-twisted Leibniz rule:

$$\nabla(fs) = f\nabla s + \lambda s \otimes df.$$ + flatness/integrability condition

automatic since C is a curve.

Hodge moduli space:

$$\begin{tikzcd} \mathfrak{M}_r^{Hod}(C) \arrow[r, "JH"] \arrow[dr, "\pi_{\mathfrak{M}}"'] & \mathcal{M}_r^{Hod}(C) \arrow[d, "\pi_{\mathcal{M}}"] \\ & \mathbb{A}^1 \end{tikzcd}$$

$$\mathfrak{M}_r^{Hod}(C) \simeq R_r^{Hod}(C) / GL_r$$ where $R_r^{Hod}(C)$

parametrizes framed λ-connections.

a $\lambda = 0$-connection is a Higgs bundle

a $\lambda = 1$-connection is an actual vector bundle w/ connection.

$\mathbb{G}_m$ acts on $\mathcal{M}_r^{Hod}(C)$, $\mathcal{M}_r^{Hod}(C)$, $\mathbb{A}^1$ is a compatible way.

$$t\cdot(\mathcal{F},\nabla,\beta) \mapsto (\mathcal{F}, t\nabla, \beta).$$

$\Rightarrow$ $R_r^{Hod}(C)|_{\mathbb{A}^1\setminus\{0\}} \longrightarrow \mathbb{A}^1\setminus\{0\}$ is trivial algebraic fibration.

<u>Cartesian squares</u>

$$\begin{tikzcd} \mathcal{M}_r^{Dol}(C) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathcal{M}_r^{Hod}(C) \arrow[d] \\ \mathcal{M}_r^{Dol}(C) \arrow[r, "\iota_{Dol}"] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathcal{M}_r^{Hod}(C) \arrow[d] \\ \{0\} \arrow[r] & \mathbb{A}^1 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathcal{M}_r^{dR}(C) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathcal{M}_r^{Hod}(C) \arrow[d] \\ \mathcal{M}_r^{dR}(C) \arrow[r, "\iota_{dR}"] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathcal{M}_r^{Hod}(C) \arrow[d] \\ \{1\} \arrow[r] & \mathbb{A}^1 \end{tikzcd}$$

<u>Simpson</u>: ⓪ NAHT gives a set-theoretic trivialization of $R_r^{Hod}(C)/\mathbb{A}^1$.

↳ counterexample shows that this is not a continuous trivialization.

① The <u>induced trivialization</u> of $\mathcal{M}_r^{Hod}(C)/\mathbb{A}^1$ is <u>topological</u>!

We can use this to prove that $(\pi_{\mathcal{M}})_* \mathbb{D}\mathbb{Q}_{\mathcal{M}_r^{Hod}(C)}$ is <u>constant</u> on $\mathbb{A}^1$.

<u>Now</u>: development of these ideas.

CoHA for the Hodge moduli space

$$\begin{tikzcd} \mathfrak{M}^{\mathrm{Hod}} \arrow[rr, "\mathrm{JH}"] \arrow[dr, "\pi_{\mathfrak{M}}"'] & & \mathcal{M}^{\mathrm{Hod}} \arrow[dl, "\pi_{\mathcal{M}}"] \\ & \mathbb{A}^1 & \end{tikzcd}$$

$\mathcal{A}^{\mathrm{Hod}}_{ii}$

Objectives: a) Defining a monoidal structure on $\mathrm{JH}_* \mathbb{D}\mathbb{Q}^{\mathrm{vir}}_{\mathfrak{M}^{\mathrm{Hod}}} \in \mathcal{D}^+_c(\mathcal{M}^{\mathrm{Hod}})$.

1st question: What monoidal structure on $\mathcal{D}^+_c(\mathcal{M}^{\mathrm{Hod}})$?

* Answer: use the morphism $\pi_{\mathcal{M}}$.

In explicit terms: $i : \mathcal{M}^{\mathrm{Hod}} \times_{\mathbb{A}^1} \mathcal{M}^{\mathrm{Hod}} \longrightarrow \mathcal{M}^{\mathrm{Hod}} \times \mathcal{M}^{\mathrm{Hod}}$

$$\mathcal{F}, \mathcal{G} \in \mathcal{D}^+_c(\mathcal{M}^{\mathrm{Hod}})$$

$$\mathcal{F} \boxdot_{\mathbb{A}^1} \mathcal{G} := i^!(\mathcal{F} \boxtimes \mathcal{G}).$$

* b) Definining a monoidal structure on $\mathcal{D}^+_c(\mathbb{A}^1)$:

$$\mathcal{F}, \mathcal{G} \in \mathcal{D}^+_c(\mathbb{A}^1) \qquad \mathcal{F} \boxdot \mathcal{G} := \mathcal{F} \overset{!}{\otimes} \mathcal{G}.$$

Prop:

$$\imath_{\mathrm{Dol}} : \mathcal{M}^{\mathrm{Dol}} \longrightarrow \mathcal{M}^{\mathrm{Hod}}$$

$$\imath_{\mathrm{dR}} : \mathcal{M}^{\mathrm{dR}} \longrightarrow \mathcal{M}^{\mathrm{Hod}}$$

$$\imath^!_{\mathrm{Dol}} : \left(\mathcal{D}^+_c(\mathcal{M}^{\mathrm{Hod}}), \boxdot_{\mathbb{A}^1}\right) \longrightarrow \left(\mathcal{D}^+_c(\mathcal{M}^{\mathrm{Dol}}), \boxdot\right) \text{ is monoidal}$$

similarly for $\imath_{\mathrm{dR}}$.

Then, the construction of the multiplication on $\mathcal{A}^{Hod}$ uses the diagram over $\mathbb{A}^1$

$$\begin{tikzcd}
 & \mathcal{E}xact^{Hod} \arrow[dl, "q"'] \arrow[dr, "p"] & \\
\mathfrak{M}^{Hod} \times \mathfrak{M}^{Hod} \arrow[d] & & \mathfrak{M}^{Hod} \arrow[d] \\
\mathcal{M}^{Hod} \times \mathcal{M}^{Hod} \arrow[rr, "\oplus"] & & \mathcal{M}^{Hod}
\end{tikzcd}$$

whose fiber over $\{0\}$ / $\{1\}$ is the diagram for Dolbeault / de Rham and is $\mathbb{C}^*$- equivariant. As before, get $m: \mathcal{A}^{Hod} \boxdot \mathcal{A}^{Hod} \to \mathcal{A}^{Hod}$ associative multiplication.

Prop:
$$i_{Dol}^!\left(\mathcal{A}^{Hod}, m\right) \simeq \left(\mathcal{A}^{Dol}, m\right)$$
$$i_{dR}\left(\mathcal{A}^{Hod}, m\right) \simeq \left(\mathcal{A}^{dR}, m\right)$$

Idea/Recall: $\mathcal{A}^{Dol}, \mathcal{A}^{dR}$ are $\simeq \mathrm{Sym}\left(BPS^{\#} \otimes H^*_{\mathbb{C}^*}\right)$ for $BPS^{\#} \in \mathrm{Perv}(\mathcal{M}^{\#})$ a Lie algebra object.

Extend this to $\mathcal{A}^{Hod}$: use relative perverse t-structures of Hansen and Scholze.

[Relative perverse t-structure]

Thm (HS) $X \xrightarrow{f} Y$ morphism of algebraic varieties

$\exists$ t-structure on $\mathcal{D}_c(X)$ s.t. $\mathcal{F} \in \mathcal{D}_c(X)$ is in $\mathcal{D}_c^{\leq 0}(X)$ / $\mathcal{D}_c^{\geq 0}(X)$ iff $\forall y \in Y$ geom point, $i_y^! \mathcal{F} \in {}^p\mathcal{D}^{\leq 0}(X_y)$ / ${}^p\mathcal{D}^{\geq 0}(X_y)$.

$\rightarrow$ interpolation of perverse t-structures on fibers.
Very natural to expect but harder to formally prove the existence! Luckily, this was done by Hansen-Scholze.

Prop. A^{Hod} is in relative perverse degrees ≥ 0
$^{p/\mathbb{A}^1}\mathcal{H}^0(A^{Hod}) =: BPS^{Hod}_{Alg} \in \mathrm{Perv}(\mathcal{M}^{Hod}/\mathbb{A}^1)$ is an algebra object.

Relative $\mathcal{IC}$ complex

Rk: $\mathcal{M}^{Hod}_r \longrightarrow \mathbb{A}^1$ is topologically trivial so

$$\mathcal{IC}(\mathcal{M}^{Hod}_r)[1] \text{ is a relative p.sheaf.}$$
$$=: \mathcal{IC}(\mathcal{M}^{Hod}_r/\mathbb{A}^1)$$

It is so that $\forall$
$$\begin{tikzcd} (\mathcal{M}^{Hod}_r)_\lambda \arrow[r, "i_\lambda"] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathcal{M}^{Hod}_r \arrow[d] \\ \{\lambda\} \arrow[r] & \mathbb{A}^1 \end{tikzcd},$$

$$i_\lambda^! \mathcal{IC}(\mathcal{M}^{Hod}_r/\mathbb{A}^1) \cong \mathcal{IC}((\mathcal{M}^{Hod}_r)_\lambda).$$

$\underline{\text{Thm}}$ (H) $\quad BPS^{Hod}_{Alg} \simeq \underset{\text{env alg}}{U}\big(\underset{\text{Lie alg}}{\underline{BPS^{Hod}_{Lie}}}\big) \in \big(\mathrm{Perv}(\mathcal{M}^{Hod}/\mathbb{A}^1), \boxdot_{/\mathbb{A}^1}\big)$

$$BPS^{Hod}_{Lie} \simeq \mathrm{Free\,Lie}_{\boxdot_{\mathbb{A}^1}}\Big(\bigoplus_{r \geq 1} \mathcal{IC}(\mathcal{M}^{Hod}_r/\mathbb{A}^1)\Big)$$

$$\mathcal{A}^{Hod} \simeq \mathrm{Sym}_{\boxdot_{\mathbb{A}^1}}\big(BPS^{Hod}_{Lie} \otimes H^*_{\mathbb{C}^*}\big).$$

$\underline{\text{Proof}}$: If we manage to do $\underline{\text{one}}$ thing, we are done thanks to all the preliminaries.

This thing is to construct a nontrivial morphism

$$\boxed{\begin{array}{c}\mathcal{IC}(\mathcal{M}^{Hod}_r/\mathbb{A}^1) \\ \downarrow \\ BPS^{Hod}_r \\ \forall r \geq 1\end{array}}$$

This morphism happens to be $\underline{\text{unique}}$ (up to rescaling).

$\underline{\text{Idea}}$: $\quad \mathfrak{M}^{Hod}_r \xrightarrow{JH} \mathcal{M}^{Hod}_r$ is generically a $\mathbb{G}_m$-gerb

So $\quad \underset{\mathcal{A}^{Hod}_r ?}{JH_* \mathbb{D}\mathbb{Q}^{vir}_{\mathfrak{M}^{Hod}_r}}\Big|_{(\mathcal{M}^{Hod}_r)^{sm}} \longleftarrow \mathcal{IC}(\mathcal{M}^{Hod}_r/\mathbb{A}^1)\Big|_{(\mathcal{M}^{Hod}_r)^{sm}}$.

needs to be extended over the singular locus.

$\underline{\text{Nontrivial}}$ since we do not know a priori that $\mathcal{A}^{Hod}_r$ is semisimple.

$\rightsquigarrow$ use MHM enhancement of everything + purity/weight estimates

Why are we done?

$$\mathcal{JE}(\mathcal{M}_r^{Hod}/\mathbb{A}^1) \longrightarrow \mathcal{BPS}_{Alg}^{Hod} \quad \forall r$$

gives $$\mathrm{Free}_{\boxdot_{\mathbb{A}^1}}\Big(\bigoplus_{r\geq 1} \mathcal{JE}(\mathcal{M}_r^{Hod}/\mathbb{A}^1)\Big) \longrightarrow \mathcal{BPS}_{Alg}^{Hod} \in \mathrm{Perv}(\mathcal{M}_r^{Hod}/\mathbb{A}^1)$$

Fact: This is an isomorphism since iso after applying $i_\lambda^! \ \forall \lambda \in \mathbb{A}^1$.

Also, $$\mathcal{BPS}_{Lie}^{Hod} \longrightarrow \mathcal{A}^{Hod} \circlearrowleft H^*(B\mathbb{C}^*) \text{ det lb on } \mathfrak{M}^{Hod}$$

$$\mathcal{BPS}_{Lie}^{Hod} \cong \mathrm{FreeLie}_{\boxdot_{/\mathbb{A}^1}}\Big(\bigoplus_{r\geq 0} \mathcal{JE}(\mathcal{M}_r^{Hod})\Big)$$

gives $$\mathcal{BPS}_{Lie}^{Hod} \otimes H(B\mathbb{C}^*) \longrightarrow \mathcal{A}^{Hod}$$

and $$\mathrm{Sym}\Big(\mathcal{BPS}_{Lie}^{Hod} \otimes H^*(B\mathbb{C}^*)\Big) \longrightarrow \mathcal{A}^{Hod}.$$

Fact: This is an iso since iso after applying $i_\lambda^! \ \forall \lambda \in \mathbb{A}^1$.

Corollary: $\pi_{\mathfrak{M}*} \mathcal{DQ}_{\mathfrak{M}^{Hod}}^{vir} \in \mathcal{D}_c^+(\mathbb{A}^1)$ is constant.

Since $\mathbb{A}^1$ is contractible, it interpolates the algebra $\mathcal{A}^{Dol}$ and $\mathcal{A}^{dR}$.

Appendix: Simpson: the space map

$$\begin{tikzcd}
R_r^{Hod}(X) \arrow[d] \arrow[dr] & \\
\mathbb{A}^{r} & (E, B)
\end{tikzcd}$$

is not continuous

Why studying this? :

* A question arising naturally since BM homologies of the stacks are iso
* Ultimate goal is to describe the algebra structure, this comparison gives 3 sides to attack the question.

↑

Want to do this since this algebra acts on the coh. of moduli of framed Higgs bundles, local systems, connections.