

Nonabelian Hodge isomorphisms for cohomological Hall algebras
 in part joint work with Davison and Ichlegel Mejia

[Hitchin, Mitsuue, Simpson, ...]

NAHT: connections between 3 types of geometric/top objects on
 a smooth projective manifold X ; later: $X = \mathbb{C}$ Smooth proj curve.
 genus ≥ 2 unless otherwise specified

① Higgs bundles \mathcal{F} v. bundle, $\theta: \mathcal{F} \rightarrow \mathcal{F} \otimes \Omega_X^1$ \mathbb{C} -linear
 on X
 Dolbeault
 vanishing Chern classes.

$$\theta(fs) = f\theta(s) \quad \forall f \text{ function}$$

s section of \mathcal{F} ,
 $\nabla \circ \nabla = 0$

semistable: $\frac{\deg(y)}{\text{rk}(y)} \leq \frac{\deg(\mathcal{F})}{\text{rk}(\mathcal{F})} \quad (\forall y \subset \mathcal{F})$ for curves.

② Local systems = representations of $\pi_1(X, x)$
 Betti

③ Connections = vector bundles with
 de Rham flat connection.

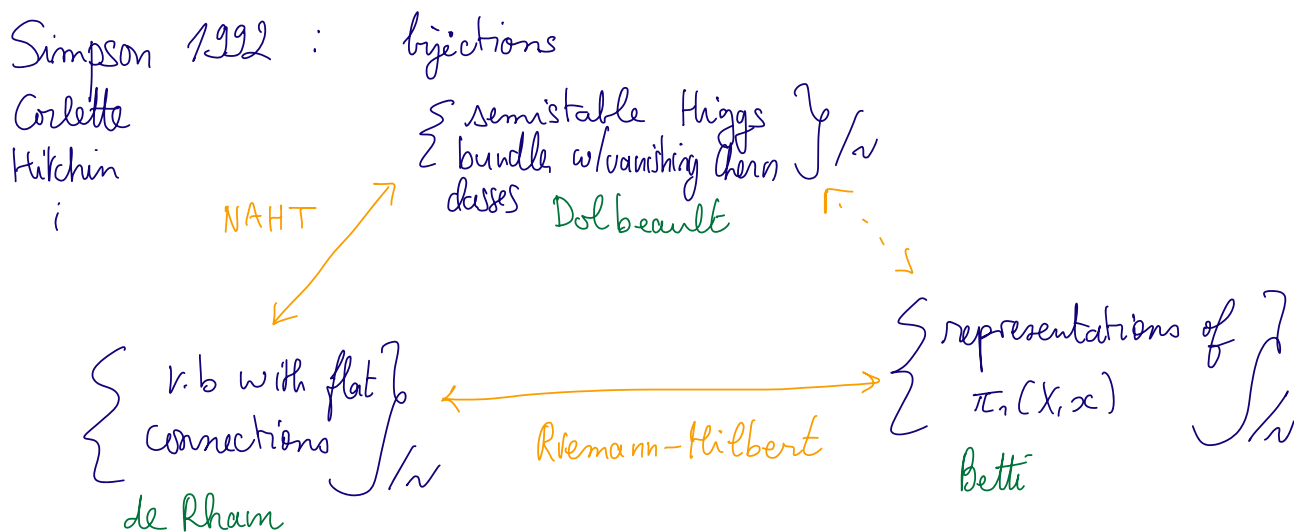
(\mathcal{F}, ∇) \mathcal{F} v. bundle,

$\nabla: \mathcal{F} \rightarrow \mathcal{F} \otimes \Omega_X^1$ satisfying
 the Leibniz rule

$$\nabla(fs) = f\nabla s + s \otimes df$$

$\forall f$ function
 s section of \mathcal{F} .

flat: $\nabla \circ \nabla = 0$



Each side has moduli. NAHT ^(also) gives comparisons between this moduli spaces.

Construction : * consider framed objects
* take the quotient by the group changing the framing.

e.g. (2) Betti side. \mathcal{L} local system of rank r on $X \ni x$ fixed framing is $\mathcal{L}_x \cong \mathbb{C}^r$.

A framed local system is $\pi_1(X, x) \longrightarrow GL_r(\mathbb{C})$

$\rightsquigarrow \text{Hom}_{\text{grp}}(\pi_1(X, x), GL_r(\mathbb{C}))$ finitely presented group

finitely presented \mathbb{C} -scheme (affine) $\parallel_{\mathbb{R}^r} \mathbb{B}$

$$X = \mathbb{C} \quad \pi_1(X, x) = \left\langle x_i, y_i \mid \prod_{i=1}^g x_i y_i x_i^{-1} y_i^{-1} = 1 \right\rangle.$$

Action of $GL_r(\mathbb{C})$ by conjugation

Betti stack: $\pi_2^B = \text{Hom}(\pi_1(X, x), GL_r(\mathbb{C})) / GL_r(\mathbb{C})$

Betti moduli space = character variety:

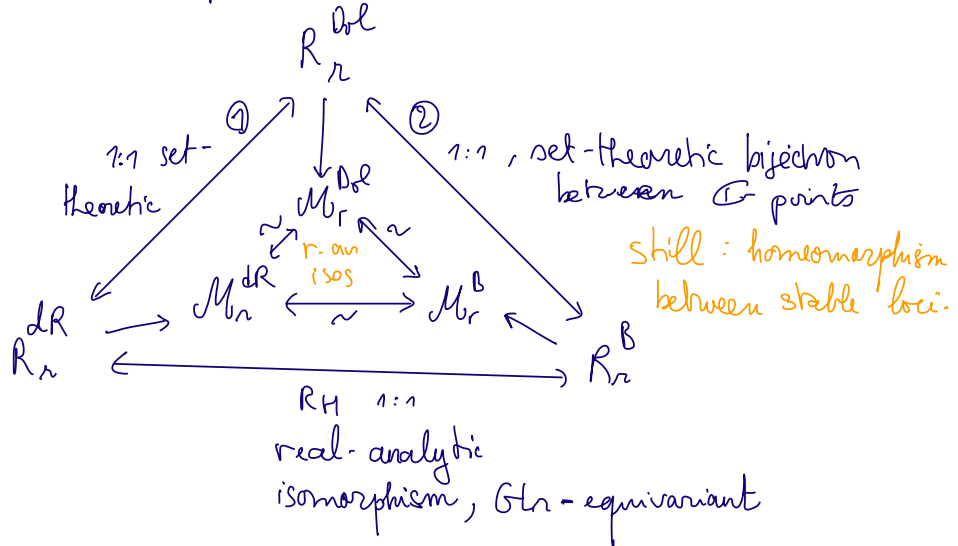
$$\mathcal{M}_r^B = \text{Hom}(\pi_1(X, x), GL_r(\mathbb{C})) // GL_r(\mathbb{C})$$

JH: $\pi_2^B \rightarrow \mathcal{M}_r^B$ Jordan-Hölder map / good moduli space.

① Dolbeault: $JH: \pi_{r,0}^{\text{Dol}}(X) \rightarrow \mathcal{M}_{r,0}^{\text{Dol}}(X)$
 $\pi_{r,0}^{\text{Dol}}(X) = \mathbb{R}_r^{\text{Dol}}(X) / GL_r$
 $\mathcal{M}_{r,0}^{\text{Dol}}(X) = \mathbb{R}_r^{\text{Dol}} // GL_r$
 $\{F, \theta, \beta: F_x \cong \mathbb{C}^r\}$

③ de Rham: $JH: \pi_r^{\text{dR}}(X) \rightarrow \mathcal{M}_r^{\text{dR}}(X)$
 $\pi_r^{\text{dR}}(X) = \mathbb{R}_r^{\text{dR}} / GL_r$
 $\mathcal{M}_r^{\text{dR}}(X) = \mathbb{R}_r^{\text{dR}} // GL_r$
 $\{F, \nabla, \beta: F_x \cong \mathbb{C}^r\}$

$\mathbb{R}_r^{\text{Dol}}(X), \mathbb{R}_r^{\text{dR}}$ are actual schemes, parametrising framed objects.
 NAHT picture [moduli space version of the previous diagram]



Simpson: The maps ①, ② are not continuous, explicit counterexample.
 Simpson's c.ex is for $X \in$ elliptic curve.

Inner triangle: we (obviously) have isos

$$H^*(\mathcal{M}_r^{\text{Dol}}) \cong H^*(\mathcal{M}_r^B) \cong H^*(\mathcal{M}_r^{\text{dR}})$$

$$H_*^{\text{BM}}(\mathcal{M}_r^{\text{Dol}}) \cong H_*^{\text{BM}}(\mathcal{M}_r^B) \cong H_*^{\text{BM}}(\mathcal{M}_r^{\text{dR}}).$$

Also, RH induces isos $H_{\text{GLr}}^*(R_r^{\text{dR}}) \cong H_{\text{GLr}}^*(R_r^B)$

Z space $H^*(Z) := H^*(X, \mathcal{O}_Z)$
 $H_*^{\text{BM}} = H_*^{-*}(Z, \mathcal{D}\mathcal{D}_Z)$
 \uparrow
 de Rham complex

$$H_{*}^{\text{BM}, \text{GLr}}(R_r^{\text{dR}}) \cong H_{*}^{\text{BM}, \text{GLr}}(R_r^B)$$

Question: Is it possible to compare H^* / H_*^{BM} for $\mathcal{M}_r^{\text{Dol}}$ and $\mathcal{M}_r^B / \mathcal{M}_r^{\text{dR}}$?

Answer: Hard question in general!

Take X a smooth projective curve

Yes: D, Davison, H, Schlegel Meijer for H_*^{BM} .

$X =$ Curve is already a highly non trivial situation.

\leadsto P=W conjecture relates filtrations on

$$H^*(\mathcal{M}_{r,d}^{\text{Dol}}(C)) \cong H^*(\mathcal{M}_{r,d}^B(C)) \quad [(\underline{r}, d) = 1]$$

New question: $H_*^{\text{BM}}(\mathcal{M}_r^{\#}(C))$ has more structure: algebra structure coming from the CoHA construction.

Can we compare the algebra structures?

Answer: Yes!

Chm: The CohAs $\bigoplus_{r \geq 0} H_*^{BM}(\mathcal{M}_r^\#(C))$ for $\# \in \{\text{Dol}, B, dR\}$ are all isomorphic.

Example: $C = \mathbb{P}^1$: $\mathbb{P}^1 \simeq S^2 \simeq \mathbb{R}^2 \cup \{\infty\}$ is simply connected and so for any $r \in \mathbb{N}$, there is a unique (up to iso) local system / flat connection of rank r .

• Also, the unique semistable Higgs bundle of degree 0 is and rank r is $\mathcal{O}_C^{\oplus r}$.

In any case, $\mathcal{M}_r^\#(C) \simeq \text{pt} / \text{GL}_r$ $\# \in \{\text{Dol}, B, dR\}$.

→ comparison of homologies/cohomologies is obvious.

What about the algebra structures?

Roughly, we want to compare the derived structures on $\mathcal{M}_r^\#(C)$.

Fact: It almost suffices to compare the tangent complexes to the derived stacks.

$$\pi_{\text{pt}} \mathcal{M}_r^\#(C) \simeq \text{pt} / \text{GL}_r \rightarrow 0 \rightarrow \text{pt} / \text{GL}_r$$

⇒ the algebra structures coincide.

by generators and relations

Going further: Describe explicitly the algebra $\bigoplus_{r \geq 0} H_*^{BM}(\mathcal{M}_r^\#(C))$.
That's a hard question!

$$\boxed{g=0} \rightsquigarrow \mathcal{U}(\mathbb{C}[x]) \simeq \text{Sym}(\mathbb{C}[x]) \simeq \bigoplus_{r \geq 0} \mathbb{C}[x_1, \dots, x_r] \text{ and product is given by } \mathbb{C}[x_1, \dots, x_r] \otimes \mathbb{C}[x_1, \dots, x_s] \rightarrow \mathbb{C}[x_1, \dots, x_{r+s}].$$

$g=1$ Already trickier. Fourier-Mukai transform is the key.

It provides isos $\mathcal{M}^{\text{Dol}} \cong \mathcal{M}_{\text{torsion}}$

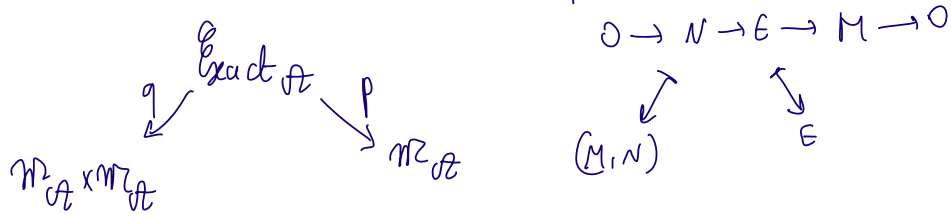
\uparrow stack of (\mathbb{P}^1, θ)
torsion $\in \text{End}(\mathbb{P}^1)$.

CoHA for torsion Higgs sheaves = $\text{CoHA}_{\text{f.l.}}(T^*C)$ is known
by work of [MMSV] in progress

related to $W_{1+\infty}$ Lie algebra.
see also [Ducrosin] for $\text{CoHA}_{\text{f.l.}}(\mathbb{A}^2)_{\text{tors}}$.
 $T^*\mathbb{A}^1$

CoHA structures [idea: Kontsevich-Sibelman, Schiffmann-Vasserot]

\mathcal{A} Abelian category with stack of objects $\mathcal{M}_{\mathcal{A}}$.
stack of short exact sequences $\text{Exact}_{\mathcal{A}}$



* p proper

* q regular enough*

Then, $m = p_* q^* : H_*^{BM}(\mathcal{M}_{\mathcal{A}}) \otimes H_*^{BM}(\mathcal{M}_{\mathcal{A}}) \rightarrow H_*^{BM}(\mathcal{M}_{\mathcal{A}})$ gives an associative algebra structure.

* q regular enough is the condition to be able to construct the pullback map

$$\text{map } \mathbb{D}\mathcal{D}_{\mathcal{M}_{\mathcal{A}} \times \mathcal{M}_{\mathcal{A}}} \rightarrow q_* \mathbb{D}\mathcal{D}_{\text{Exact}_{\mathcal{A}}} [\text{shift}]$$

q "quasi-smooth": q comes from a map between derived stacks

$$q : \text{Exact}_{\mathcal{A}} \rightarrow \mathcal{M}_{\mathcal{A}} \times \mathcal{M}_{\mathcal{A}} \text{ s.t. } \mathbb{T}q \text{ is perfect in degrees } \leq 1$$

in this case, shift = \pm virtual rank of $\mathbb{T}q$
virtual relative dimension of q .

for us: homological dimension of \mathcal{A} is

- 0 $\Rightarrow H_*^{BM}(\mathcal{M}_{\mathcal{A}})$ is a shuffle algebra
- 1 $\Rightarrow H_*^{BM}(\mathcal{M}_{\mathcal{A}})$ is a "Yangian"
- 2 $\Rightarrow H_*^{BM}(\mathcal{M}_{\mathcal{A}})$ is a "Yangian"

i.e. $\cong U(\mathfrak{g}[u])$
after taking associated graded Lie algebras of polynomials w/ coefficients.

Possible to work over $\mathcal{M}_{\mathcal{A}}$

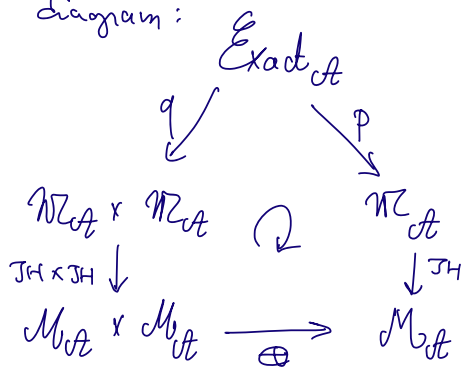
good moduli space
Alper-Hall-Knight
behaves like a
GIT quotient

$$\mathcal{M}_{\mathcal{A}} \xrightarrow{\text{JH}} \mathcal{M}_{\mathcal{A}} \xrightarrow{p} \text{pt}$$

$$A := \text{JH}_* \mathbb{D}\mathcal{D}_{\mathcal{M}_{\mathcal{A}}}^{\text{vir}} \in \mathcal{D}_c^+(\mathcal{M}_{\mathcal{A}})$$

$$p_* A = H_*^{BM}(\mathcal{M}_{\mathcal{A}})$$

Use the diagram:



(Davis) \mathcal{A} 2CY Abelian category.

BPS algebra Prop \mathcal{A} is concentrated in nonnegative perverse degrees: $\text{PH}^i(\mathcal{A}) = 0$ if $i < 0$

$\Rightarrow \text{PH}^0(\mathcal{A}) =: \text{BPJ}_{\mathcal{A}, \text{Alg}} \in \text{Per}(\mathcal{M}_{\mathcal{A}})$ is an associative algebra object.

Crucial in the comparison of CoHAs for NAHT

Theorem (DHS) Let \mathcal{A} be a totally negative^{2CY} Abelian category.

- $\mathcal{M}_{\mathcal{A}}$ moduli stack of objects
- $\mathcal{JH}: \mathcal{M}_{\mathcal{A}} \rightarrow \mathcal{M}_{\mathcal{A}}$ good moduli space
- $\text{Ext}^{2i}(M, N) \cong \text{Ext}^i(N, M)^*$ functorially (2CY)
- $\chi(M, N) = \sum_{i \in \mathbb{Z}} (-1)^i \text{ext}^i(M, N) < 0$ if $M, N \neq 0$ (ht. negative)

$\Sigma_{\mathcal{A}} \subset \pi_0(\mathcal{M}_{\mathcal{A}})$ set of conn. components containing a simple rep of \mathcal{A} .

Then: $\text{BPJ}_{\mathcal{A}, \text{Alg}} \cong \bigcup_{\text{env. alg}} (\text{BPJ}_{\mathcal{A}, \text{Lie}})$
 Lie alg object in $\text{Per}(\mathcal{M}_{\mathcal{A}})$

where $\mathcal{BPP}_{\mathcal{A}, \text{lie}} := \text{FreeLie}_{\square} \left(\bigoplus_{a \in \mathcal{E}_{\mathcal{A}}} \mathcal{I}^{\circ}(M_{\mathcal{A}, a}) \right)$.

Free Lie algebra.

and (PBW-type iso)

$$\mathcal{A}_{\mathcal{A}} \cong \text{Sym}_{\square} \left(\mathcal{BPP}_{\mathcal{A}, \text{lie}} \otimes H_{\mathbb{C}}^* \right)$$

as constructible complexes.

Proof: hard, not the subject of this talk.

examples: * $\text{Rep } \pi_1(\mathbb{C}, x)$

* $\text{Conn}(\mathbb{C})$

$\mu \in \mathbb{R}$
slope * $\text{Higgs}_{\mu}^{\text{ss}}(\mathbb{C})$

$g(\mathbb{C}) \geq 2$

CottAs defined by
Dariusson, Pinta-Saker,
Sala-Schiffmann
in the non-sheafed setting.

* Euler form: $(F, Y) = 2(1-g) \text{rk}(F) \text{rk}(Y)$ in all
three cases (\Rightarrow totally negative)

* $\pi_0(\pi_1^{\#}) \cong \mathbb{N}$

* $\sum_{\mathcal{A}} = \mathbb{N} \setminus \{0\}$ in all 3 cases since we have simple
objects in all ranks ($g(\mathbb{C}) \geq 2$).

Goal: Theorem: $H_{*}^{BM}(\mathbb{R}^{\#})$ $\# \in \{Dol, Betti, dR\}$ have algebra structures and they are all isomorphic through NHT.

Idea of proof: * The derived Riemann-Hilbert correspondence gives an isomorphism of derived analytic stacks

$$\mathcal{M}_r^{dR}(C) \simeq \mathcal{M}_r^B(C)$$

and the derived structure knows how the GHA multiplication is constructed: this is enough.

* Hodge moduli stack $\mathcal{M}_r^{Hod}(C)$ parametrising λ -connections
 \downarrow
 A^1

w/ varying λ .

Recall: a λ -connection is (\mathcal{F}, ∇) where \mathcal{F} is a v.b. on C and $\nabla: \mathcal{F} \rightarrow \mathcal{F} \otimes \Omega_C^1$ satisfies the λ -twisted Leibniz rule:

$$\nabla(fs) = f \nabla s + \lambda s \otimes df. \quad + \text{flatness/integrability condition}$$

automatic since C is a curve.

Hodge moduli space:

$$\begin{array}{ccc} \mathcal{M}_r^{Hod}(C) & \xrightarrow{JH} & \mathcal{M}_r^{Hod}(C) \\ \pi_{\mathcal{M}_r} \downarrow & & \downarrow \pi_{\mathcal{M}_r} \\ & A^1 & \end{array}$$

$$\mathcal{M}_r^{Hod}(C) \simeq R_r^{Hod}(C) / G_r \quad \text{where } R_r^{Hod}(C)$$

parametrizes framed λ -connections.

a $\lambda=0$ -connection is a Higgs bundle

a $\lambda=1$ -connection is an actual vector bundle w/ connection.

G_m acts on $\pi_n^{\text{Hod}}(C)$, $\mathcal{M}_n^{\text{Hod}}(C)$, A^1 is a compatible way.

$$t \cdot (\mathcal{F}, \nabla, \beta) \mapsto (\mathcal{F}, t\nabla, \beta).$$

$\Rightarrow R_n^{\text{Hod}}(C) \Big|_{A^1(\mathbb{C})} \rightarrow A^1(\mathbb{C})$ is trivial algebraic fibration.

Cartesian squares

$$\begin{array}{ccc} \pi_n^{\text{Dol}}(C) & \rightarrow & \pi_n^{\text{Hod}}(C) \\ \downarrow & \lrcorner & \downarrow \\ \mathcal{M}_n^{\text{Dol}}(C) & \xrightarrow{\text{sol}} & \mathcal{M}_n^{\text{Hod}}(C) \\ \downarrow & \lrcorner & \downarrow \\ \{0\} & \rightarrow & A^1 \end{array} \quad \text{and} \quad \begin{array}{ccc} \pi_n^{\text{dR}}(C) & \rightarrow & \pi_n^{\text{Hod}}(C) \\ \downarrow & \lrcorner & \downarrow \\ \mathcal{M}_n^{\text{dR}}(C) & \xrightarrow{\text{zDR}} & \mathcal{M}_n^{\text{Hod}}(C) \\ \downarrow & \lrcorner & \downarrow \\ \{1\} & \rightarrow & A^1 \end{array}$$

Simpson: ① NAHT gives a set-theoretic trivialization of $R_n^{\text{Hod}}(C)/A^1$.
 \hookrightarrow counterexample shows that this is not a continuous trivialization.

① The induced trivialization of $\mathcal{M}_n^{\text{Hod}}(C)/A^1$ is topological!

We can use this to prove that $(\pi_n \pi_*) \otimes \mathbb{R} \mathcal{M}_n^{\text{Hod}}(C)$ is constant on A^1 .

Now: development of these ideas.

CoHA for the Hodge moduli space

$$\begin{array}{ccc} \mathcal{M}^{\text{Hod}} & \xrightarrow{\text{JH}} & \mathcal{M}^{\text{Hod}} \\ \pi_{\mathcal{M}} \downarrow & & \downarrow \pi_{\mathcal{M}} \\ \mathbb{A}^1 & & \mathbb{A}^1 \end{array}$$

$$\mathcal{F}_{ii}^{\text{Hod}}$$

Objectives: a) Defining a monoidal structure on $\text{JH}^* \text{DR}_{\pi_{\mathcal{M}}^{\text{Hod}}}^{\text{int}} \in \mathcal{D}_c^+(\mathcal{M}^{\text{Hod}})$.
 1st question: What monoidal structure on $\mathcal{D}_c^+(\mathcal{M}^{\text{Hod}})$?

* Answer: use the morphism $\pi_{\mathcal{M}}$.

In explicit terms: $i: \mathcal{M}^{\text{Hod}} \times_{\mathbb{A}^1} \mathcal{M}^{\text{Hod}} \rightarrow \mathcal{M}^{\text{Hod}} \times \mathcal{M}^{\text{Hod}}$

$$\mathcal{F}, \mathcal{G} \in \mathcal{D}_c^+(\mathcal{M}^{\text{Hod}})$$

$$\mathcal{F} \boxtimes_{\mathbb{A}^1} \mathcal{G} := i^!(\mathcal{F} \boxtimes \mathcal{G}).$$

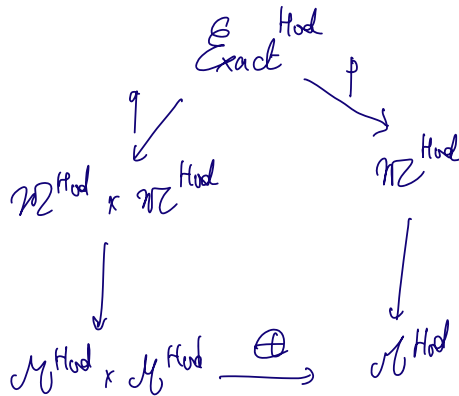
* b) Defining a monoidal structure on $\mathcal{D}_c^+(\mathbb{A}^1)$:

$$\mathcal{F}, \mathcal{G} \in \mathcal{D}_c^+(\mathbb{A}^1) \quad \mathcal{F} \boxtimes \mathcal{G} := \mathcal{F} \overset{!}{\otimes} \mathcal{G}.$$

Prop: $r_{\text{Dol}}: \mathcal{M}^{\text{Dol}} \rightarrow \mathcal{M}^{\text{Hod}}$
 $r_{\text{dR}}: \mathcal{M}^{\text{dR}} \rightarrow \mathcal{M}^{\text{Hod}}$

$r_{\text{Dol}}^!: (\mathcal{D}_c^+(\mathcal{M}^{\text{Hod}}), \boxtimes_{\mathbb{A}^1}) \rightarrow (\mathcal{D}_c^+(\mathcal{M}^{\text{Dol}}), \boxtimes)$ is monoidal
 [similarly for r_{dR} .

Then, the construction of the multiplication on \mathcal{A}^{Hod} uses the diagram over A^1



whose fiber over $\{0\}$ is the diagram for Dolbeault
 $\{1\}$ and is \mathbb{C}^* -equivariant. As before, get $m: \mathcal{A}^{\text{Hod}} \boxtimes \mathcal{A}^{\text{Hod}} \rightarrow \mathcal{A}^{\text{Hod}}$
 associative multiplication.

Prop: $i_{\text{Dol}}^! (\mathcal{A}^{\text{Hod}}, m) \simeq (\mathcal{A}^{\text{Dol}}, m)$

$i_{\text{dR}}^! (\mathcal{A}^{\text{Hod}}, m) \simeq (\mathcal{A}^{\text{dR}}, m)$

Take all $\mathcal{A}^{\text{Dol}}, \mathcal{A}^{\text{dR}}$ are $\simeq \text{Sym}(\text{BPJ}^{\#} \otimes H_{\mathbb{C}^*}^*)$ for
 $\text{BPJ}^{\#} \in \text{Per}(\mathcal{A}^{\#})$ a Lie algebra object.

Extend this to \mathcal{A}^{Hod} : use relative perverse t-structures of
 Hansen and Scholze.

[Relative perverse t-structure]

Thm (HS) $X \xrightarrow{f} Y$ morphism of algebraic varieties

\exists t-structure on $\mathcal{D}_{\mathbb{C}}(X)$ s.t. $\mathcal{F} \in \mathcal{D}_{\mathbb{C}}(X)$ is in

$\mathcal{D}_{\mathbb{C}}^{\leq 0}(X)$ iff $\forall y \in Y$, $i_{y!} \mathcal{F} \in \mathcal{D}_{\mathbb{C}}^{\leq 0}(X_y)$
 $\mathcal{D}_{\mathbb{C}}^{\geq 0}(X)$ iff $\forall y \in Y$, $i_{y!} \mathcal{F} \in \mathcal{D}_{\mathbb{C}}^{\geq 0}(X_y)$

\hookrightarrow interpolation of perverse t -structures on fibers.
 Very natural to expect but harder to formally prove the existence! Luckily, this was done by Hansen-Scholze.

Prop. A^{Hod} is in relative perverse degrees ≥ 0
 $p/A^1 \mathcal{H}^0(A^{\text{Hod}}) =: \mathcal{B}P_{\text{Alg}}^{\text{Hod}} \in \text{Perv}(\mathcal{M}^{\text{Hod}}/A^1)$ is
 an algebra object.

Relative $\mathcal{D}\mathcal{E}$ complex

Rk. $\mathcal{M}_{b_2}^{\text{Hod}} \rightarrow A^1$ is topologically trivial so

$\mathcal{D}\mathcal{E}(\mathcal{M}_{b_2}^{\text{Hod}})[1]$ is a relative p-sheaf.
 $\therefore \mathcal{D}\mathcal{E}(\mathcal{M}_{b_2}^{\text{Hod}}/A^1)$

It is so that \forall

$$\begin{array}{ccc}
 (\mathcal{M}_{b_2}^{\text{Hod}})_\lambda & \xrightarrow{\quad \iota_\lambda \quad} & \mathcal{M}_{b_2}^{\text{Hod}} \\
 \downarrow & \lrcorner & \downarrow \\
 \{ \lambda \} & \longrightarrow & A^1
 \end{array}$$

$$\iota_\lambda! \mathcal{D}\mathcal{E}(\mathcal{M}_{b_2}^{\text{Hod}}/A^1) \cong \mathcal{D}\mathcal{E}((\mathcal{M}_{b_2}^{\text{Hod}})_\lambda)$$

$$\begin{aligned}
 \mathcal{E}hm(H) \quad \mathcal{B}P\mathcal{J}_{Alg}^{Hod} &\cong \bigcup_{\substack{env \\ alg}} \left(\mathcal{B}P\mathcal{J}_{lie}^{Hod} \right) \in \left(\text{Per}(\mathcal{M}^{Hod}/\mathbb{A}^1), \square_{\mathbb{A}^1} \right) \\
 \mathcal{B}P\mathcal{J}_{lie}^{Hod} &\cong \text{Free}_{lie, \square_{\mathbb{A}^1}} \left(\bigoplus_{r \geq 1} \mathcal{J}\mathcal{E}(\mathcal{M}_r^{Hod}/\mathbb{A}^1) \right) \\
 \mathcal{A}^{Hod} &\cong \text{Sym}_{\square_{\mathbb{A}^1}} \left(\mathcal{B}P\mathcal{J}_{lie}^{Hod} \otimes H_{\mathbb{C}}^* \right).
 \end{aligned}$$

Proof: If we manage to do one thing, we are done thanks to all the preliminaries.

This thing is to construct a nontrivial morphism. This morphism happens to be unique (upto rescaling).

$$\begin{array}{c}
 \mathcal{J}\mathcal{E}(\mathcal{M}_r^{Hod}/\mathbb{A}^1) \\
 \downarrow \mathcal{B}P\mathcal{J}_r^{Hod} \\
 \cdot \quad r \geq 1
 \end{array}$$

Idea: $\mathcal{M}_r^{Hod} \xrightarrow{JH} \mathcal{M}_r^{Hod}$ is generically a G_m -gib.

$$\text{So } JH \# \mathcal{D}_{\mathcal{M}_r^{Hod}}^{vir} \leftarrow \mathcal{J}\mathcal{E}(\mathcal{M}_r^{Hod}/\mathbb{A}^1) \Big|_{(\mathcal{M}_r^{Hod})^{sm}}.$$

needs to be extended over the singular locus.

Nontrivial since we do not know a priori that \mathcal{A}_r^{Hod} is

semisimple.

→ use MHM enhancement of everything + purity/weight estimates

Why are we done?

$$\mathcal{D}E(\mathcal{M}_r^{\text{Hod}}/A^1) \rightarrow \text{BPJ}_{\text{Alg}}^{\text{Hod}} \quad \forall r$$

gives $\text{Free}_{A^1} \left(\bigoplus_{r \geq 1} \mathcal{D}E(\mathcal{M}_r^{\text{Hod}}/A^1) \right) \rightarrow \text{BPJ}_{\text{Alg}}^{\text{Hod}} \in \text{Per}(\mathcal{M}_r^{\text{Hod}}/A^1)$

Fact: This is an isomorphism since iso after applying $i_! \quad \forall \lambda \in A^1$.

Also, $\text{BPJ}_{\text{lie}}^{\text{Hod}} \rightarrow A^{\text{Hod}} \supset H^*(BC^*)$ let lb on $\mathcal{M}_r^{\text{Hod}}$

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 $\text{Free}_{A^1} \left(\bigoplus_{r \geq 0} \mathcal{D}E(\mathcal{M}_r^{\text{Hod}}) \right)$

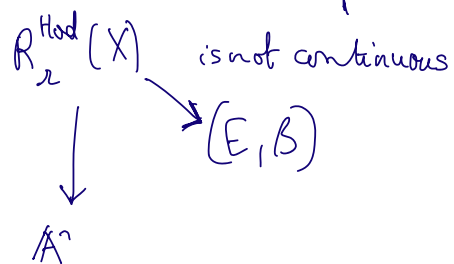
gives $\text{BPJ}_{\text{lie}}^{\text{Hod}} \otimes H^*(BC^*) \rightarrow A^{\text{Hod}}$

and $\text{Sym} \left(\text{BPJ}_{\text{lie}}^{\text{Hod}} \otimes H^*(BC^*) \right) \rightarrow A^{\text{Hod}}$.

Fact: This is an iso since iso after applying $i_! \quad \forall \lambda \in A^1$.

Corollary: $\pi_{2r} * \mathcal{DQ}_{\mathcal{M}_r^{\text{Hod}}}^{\text{vir}} \in \mathcal{D}_c^+(A^1)$ is constant.
 Since A^1 is contractible, it interpolates the algebra \mathcal{A}^{dR} and \mathcal{A}^{dR} .

Appendix: Simpson: the space map



Why studying this?

- * A question arising naturally since BM homologies of the stacks are iso
- * Ultimate goal is to describe the algebra structure, this comparison gives 3 sides to attack the question.

↑

want to do this since this algebra acts on the coh. of module of framed Higgs bundles, local systems, connections.