Cohomological integrality for symmetric quotient stacks

1- Situation and context
2- Motivation
3- Operations
4- Integrality theorem
[5- More motivation from 3 Calabi-Yau categories]
6- Examples
7- Strengthening
8- Construction of P_λ

References: arXiv: 2406.09218 $\rightsquigarrow$ could have been understood 15 years ago.
arXiv: 2408.15786

On arXiv yesterday:

Markus Reineke, Donaldson-Thomas invariants of symmetric quivers: quick overview.

Today: better title: Donaldson-Thomas invariants of symmetric representations of reductive groups.

idea: generalizing CoHDT from mod stacks of objects in some categories to some stacks.

Keywords: BPS state counts

We work over $\mathbb{C}$

1 - Situation

$G = GL_n(\mathbb{C})$, $SL_n(\mathbb{C})$, $(\mathbb{C}^*)^N$, $Sp_{2n}(\mathbb{C})$, ...

More generally, G: reductive group (unipotent radical is trivial)

$\underset{\text{char } 0}{=}$ linearly reductive

(finite-dimensional representations are semisimple)

<u>non-example</u>: $G = \mathbb{G}_a$ additive group

acts on $V = \mathbb{C}^2$ via $\mathbb{G}_a \hookrightarrow \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\}$.

V is non-trivial extension of $\mathbb{C}$ by $\mathbb{C}$.

* $T \subset G$ maximal torus. $T \cong (\mathbb{C}^*)^{\mathrm{rank}(G)}$

e.g. $\mathrm{diag} \simeq (\mathbb{C}^*)^n \subset GL_n(\mathbb{C})$.

* representation: $G \longrightarrow GL(V)$, V $\mathbb{C}$ vector space, finite-dimensional.

$GL_2(\mathbb{C})$, $SL_2(\mathbb{C}) \curvearrowright \mathbb{C}^2$.

Characters: $X^*(T) = \{\alpha : T \to \mathbb{G}_m\} \cong \mathbb{Z}^{\mathrm{rk}\, G}$

Cocharacters $X_*(T) = \{\lambda : \mathbb{G}_m \to T\} \cong \mathbb{Z}^{\mathrm{rk}\, G}$

Pairing $\alpha \circ \lambda : \mathbb{G}_m \longrightarrow \mathbb{G}_m$

$$z \longmapsto z^{\langle \lambda, \alpha \rangle}$$

$$\langle -,- \rangle : X_*(T) \times X^*(T) \to \mathbb{Z}.$$

Weights $T \curvearrowright V$ diagonalizable:

$$V = \bigoplus_{\alpha \in X^*(T)} V_\alpha$$

$$V_\alpha = \{ v \in V \mid t \cdot v = \alpha(t) \cdot v \ \ \forall t \in T \}$$

$$\mathcal{W}(V) = \{ \alpha \in X^*(T) \mid V_\alpha \neq 0 \} \text{ weights of } V.$$

In particular, $\mathcal{W}(\mathfrak{g})$ weights of $\mathfrak{g} = \mathrm{Lie}(G)$.

ex. $GL_2(\mathbb{C}) \curvearrowright \mathbb{C}^2 = \mathbb{C}e_1 \oplus \mathbb{C}e_2$

$\cup \qquad (1,0) \qquad (0,1)$

$(\mathbb{C}^*)^2$

$$(t_1, t_2) e_1 = t_1 e_1$$
$$(t_1, t_2) e_2 = t_2 e_2$$

V symmetric: $\dim V_\alpha = \dim V_{-\alpha} \quad \forall \alpha \in X^*(T)$

$\Leftrightarrow$ $V \cong V^*$ (a representation is determined by its character)

sort of weakening of symplecticity, appears sometimes when def Coulomb branches.

ex: $\cdot\ T^*V = V \oplus V^*$, V rep of G

$\cdot$ any V rep of $SL_2(\mathbb{C})$

$\cdot$ $\mathfrak{g}$ adjoint of G

$O(2n+1)$ $Sp(2n)$

$\cdot$ any representations in type B_n, C_n, E_7, E_8, F_2

Weyl group $\quad W = N_G(T)/T$

$$N_G(T) = \{g \in G \mid gTg^{-1} = T\}$$

$$W_{GL_n} \cong W_{SL_n} \cong \mathfrak{S}_n \text{ symmetric group.}$$

In general: W is a Coxeter group.

T torus: $W_T = \{e\}$

$W \curvearrowright$ weights of V $= \mathcal{W}(V)$.

Cohomological integrality

$H_G^*(V)$ equivariant cohomology

V v.space $\Rightarrow$ contractible

$$H^*_G(V) \cong H^*_G(pt) \cong H^*(BG)$$

E_G contractible space with free G action.

$BG = EG/_G$.

ex: $H^*_{\mathbb{C}^*}(pt)$

$G = \mathbb{C}^* \curvearrowright \mathbb{C}^N \setminus \{0\}$ free

$$\mathbb{C}^\infty \setminus \{0\} = \bigcup_N \mathbb{C}^N \setminus \{0\}$$

$$\mathbb{P}^\infty = \mathbb{C}^\infty \setminus \{0\} / \mathbb{C}^*$$

$$H^*(\mathbb{P}^N) \cong \mathbb{Q}[x]/(x^{N+1}) \qquad \deg(x) = 2$$

$$H^*(\mathbb{P}^\infty) = H^*_{\mathbb{C}^*}(pt) \cong \mathbb{Q}[x]$$

$G = T = (\mathbb{C}^*)^n$ $\qquad H^*_T(pt) \cong \mathbb{Q}[x_1, \dots, x_n]$

G general $\qquad H^*_G(pt) \cong H^*_T(pt)^W \qquad T \subset G$ max torus.

$\rightsquigarrow H^*_G(pt) \cong \mathbb{Q}[x_1, \dots, x_n]^{\mathfrak{S}_n}$ symmetric polynomials

$G = GL_2(\mathbb{C}) \qquad \mathbb{Q}[x_1 + x_2, x_1 x_2]$

In general $H^*_G(pt)$ is a polynomial algebra

in particular, $\dim_{\mathbb{Q}} H^*_G(pt) = +\infty$.

Cohomological integrality

Extract a finite-dimensional subspace

$$P_0 \subset H^*(V/G)$$

that generates (in the sense of parabolic induction)

$P_0 =$ " cuspidal cohomology " of V/G.

$\hookrightarrow$ analogy with $\begin{cases} \text{character sheaves (rep of fin grp of Lie type)} \\ \text{Hecke eigensheaves (Langlands)} \end{cases}$

2- Context and motivation

ⓐ Topology of the action of G on V (= of the quotient stack V/G)

of the GIT quotient $V/\!/G \overset{def}{=} \mathrm{Spec}(\mathbb{C}[V]^G)$

finite type affine scheme (Hilbert)

$V/\!/G$ classifies closed G-orbits in V.

ex: ① $\mathbb{C}^* \circlearrowright \mathbb{C}^N$ poids 1 $\quad \mathbb{C}^N/\!/\mathbb{C}^* = pt$

② $\mathbb{C}^* \circlearrowright \mathbb{C}^2 \quad t\cdot(u,v) = (tu, t^{-1}v)$

$\lambda \neq 0 \quad \{xy = \lambda\}$ are the closed orbits

$\{0\}$

$\rightsquigarrow \mathbb{C}^2/\!/\mathbb{C}^* \cong \mathbb{C}$

$\mathbb{C}[x,y]^{\mathbb{C}^*} \cong \mathbb{C}[xy] \subset \mathbb{C}[x,y].$

③ $G \curvearrowright \mathfrak{g}$ adjoint rep.

$\mathfrak{g}/\!/G \cong \mathfrak{t}/\!/W \qquad \mathfrak{t} = \mathrm{Lie}\, T$

$\cong \mathbb{A}^{\mathrm{rk}\, G}$

④ non smooth:

$\mathbb{C}^* \curvearrowright \mathbb{C}^4 \quad t\cdot(u,v,w,x) = (tu, tv, t^{-1}w, t^{-1}x)$

$\mathbb{C}^4/\!/\mathbb{C}^* \cong \mathrm{Spec}\left(\mathbb{C}[ac, ad, bc, bd]\right)$

$\cong \mathrm{Spec}\left(\mathbb{C}[A,B,C,D]/\langle AD = BC\rangle\right)$

Computing generators of $\mathbb{C}[V]^G$: difficult and old problem of invariant theory, even for $SL_2(\mathbb{C})$ [invariants of binary forms]

Sylvester - Franklin 1879 deg ≤ 10 with mistakes

von Gall 1880, Shioda, 1967

Brouwer - Popoviciu 2010: deg 9 92 generators

deg 10 104 gens.

deg 11: not much known

Interesting names for some invariants

catalecticant: deg $\frac{n}{2}+1$ inv for binary forms of even degree

canonizant deg $\frac{n+1}{2}$ inv for binary forms of odd degree.

? Hilbert series of $\mathbb{C}[V]^G$

$$H(V,G) = \sum_{d\in\mathbb{N}}' \dim \mathbb{C}[V]^G_{\deg = d}\, t^d = ?$$

Formula for HS of $\mathbb{C}[\mu^{-1}(0)]^G$, μ moment map, for symplectic singularities

Cohomological integrality conjecture $\rightsquigarrow$ algorithmic computation of

$IH^*(V/\!/G)$

$IH(X) = \left\{ \text{intersection cohomology} = \left\{ \begin{array}{l} \text{singular cohomology if } X \text{ smooth} \\ \text{encodes information regarding singularities otherwise.} \end{array} \right. \right.$

(b) Topology of $\mathfrak{M} \longrightarrow \mathcal{M}$

$\mathfrak{M}$: smooth Artin stack

$\mathcal{M}$: good moduli space (Alper)

loc-global principle

specialisation

globalisation of $V/G \longrightarrow V/\!/G$.

étale slices (Luna, Alper-Hall-Rydh)

ex $\mathfrak{M} = \mathrm{Bun}_G(C)$

(c) Introducing and studying new enumerative invariants for (G,V), $V/\!/G$, $\mu^{-1}(0)/\!/G$ = Higgs branches

③ Operations

Parabolic induction

V representation of G

$\lambda : \mathbb{G}_m \to T$ cocharacter

$$G^\lambda = \{ g \in G \mid \lambda(t) g \lambda(t)^{-1} = g \quad \forall t \in \mathbb{C}^* \}$$

$\subset G$ levi subgroup

<u>Note</u> G^λ reductive

$T \subset G^\lambda$.

$$V^\lambda = \{ v \in V \mid \lambda(t) v \lambda(t)^{-1} = v \quad \forall t \in \mathbb{C}^* \}$$

$\subset V$ subspace

$$G^{\lambda \geq 0} = \{ g \in G \mid \lim_{t \to 0} \lambda(t) g \lambda(t)^{-1} \text{ exists} \}$$

$\subset G$ parabolic subgroup

$$V^{\lambda \geq 0} = \{ v \in V \mid \lim_{t \to 0} \lambda(t) \cdot v \text{ exists} \}$$

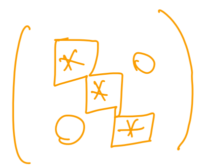
$\subset V$

subspace

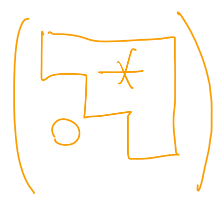
$G = GL_n \qquad V = T^*\mathbb{C}^n$

$\mathbb{G}_m \longrightarrow GL_n$

$t \longmapsto \begin{pmatrix} t^2 & & 0 \\ & t & \\ 0 & & 1 \end{pmatrix}$



$V^\lambda = T^* \begin{pmatrix} 0 \\ 0 \\ * \end{pmatrix}$



$V^{\lambda \geq 0} = \mathbb{C}^n \oplus \begin{pmatrix} 0 \\ 0 \\ * \end{pmatrix}$

Induction diagram

$$\begin{tikzcd}[column sep=small]
 & V^{\lambda \geq 0}/G^{\lambda\geq 0} \arrow[dl, "q_\lambda"', "\text{smooth}"] \arrow[dr, "p_\lambda", "\text{proper}"'] & \\
V^\lambda/G^\lambda & & V/G
\end{tikzcd}$$

$$\mathrm{Ind}_\lambda := p_\lambda^* q_\lambda^* : H^*(V^\lambda/G^\lambda) \longrightarrow H^*(V/G)$$

parabolic induction

$$\mathrm{Ind}_\lambda : \mathbb{Q}[x_1, \dots, x_r]^{W^\lambda} \longrightarrow \mathbb{Q}[x_1, \dots, x_r]^W.$$

$\exists$ translation of coh degree making Ind_λ graded.

Explicit formula:

$\alpha \in X^*(T)$ may be seen as an element of $H_T^*(pt) \cong \mathrm{Sym}(\mathfrak{t}^*)$

$\alpha : T \to \mathbb{G}_m \quad d\alpha(1) : \mathfrak{t} \to \mathbb{C}$.
$\in \mathfrak{t}^*$

$$k_\lambda := \frac{\displaystyle\prod_{\substack{\alpha \in \mathcal{W}(V) \\ \langle \lambda, \alpha \rangle > 0}} \alpha^{\dim V_\alpha}}{\displaystyle\prod_{\substack{\alpha \in \mathcal{W}(\mathfrak{g}) \\ \langle \lambda, \alpha \rangle > 0}} \alpha^{\dim \mathfrak{g}_\alpha}} \in \mathrm{Frac}\left(H_T^*(pt)\right)$$

$$\mathrm{Ind}_\lambda(f) = \frac{1}{|W^\lambda|} \sum_{w \in W} w \cdot (f k_\lambda).$$

Proof: Calculation after localization and computation of Euler class, using Borel-Weil-Bott Thm.

Tautological classes

$K \subset G$ normal subgroup

$$H^*_G(pt) \cong H^*_{G/K}(pt) \otimes H^*_K(pt)$$

non-canonical.

$\rightsquigarrow$ action of $H^*_K(pt)$ on $H^*_G(pt)$.

Cohomological integrality theorem

$$\lambda \sim \mu \iff \begin{cases} G^\lambda = G^\mu \\ V^\lambda = V^\mu \end{cases}$$

$\rightsquigarrow \mathcal{P}_V = X_*(T)/\sim$ finite set

$$\begin{array}{c} \mathcal{P}_V \\ \circlearrowleft \\ W \end{array}$$

$G_\lambda = \ker(G^\lambda \to GL(V^\lambda)) \cap Z(G^\lambda) \subset G$ normal subgroup.

$W_\lambda = \{w \in W \mid w \cdot \lambda \sim \lambda\} \subset W$ subgroup

$\varepsilon_{V,\lambda} : W_\lambda \to \{\pm 1\}$ such that

$$k_{w\cdot\lambda} = \varepsilon_{V,\lambda}(w)\, k_\lambda \quad \forall w \in W_\lambda.$$

Thm (H, 2024) Let V be a symmetric (self-dual) representation of G.
For $\lambda \in X_*(T)$, $\exists\, P_\lambda \subset H^*_{G^\lambda}(V^\lambda)$ finite-dimensional and graded, stable under the W_λ-action, s.t.

isotypic component

$$\bigoplus_{\tilde{\lambda} \in \mathcal{P}_V/W} \left(P_\lambda \otimes H^*(pt/G_\lambda)\right)^{\varepsilon_{V,\lambda}} \xrightarrow[\oplus \mathrm{Ind}_\lambda]{} H^*_G(V)$$

is a graded isomorphism + P_λ determined by the existence of such an isomorphism.

Def $p_{\lambda,i} = \dim P_\lambda^i \in \mathbb{N}$ "refined DT invariants of (G,V)".

new enumerative invariants we seek to understand and interpret geometrically.

5- Examples

① $\underset{G}{GL_2(\mathbb{C})} \curvearrowright \underset{V}{(T^*\mathbb{C}^2)^g}$ $\quad g \geq 0 \qquad T = (\mathbb{C}^*)^2 \subset GL_2(\mathbb{C})$

$$\lambda_0 : \mathbb{G}_m \longrightarrow T$$
$$t \longmapsto 1$$

$$\lambda_1 : \mathbb{G}_m \longrightarrow T$$
$$t \longmapsto (t,1)$$

$$\lambda_2 : \mathbb{G}_m \longrightarrow T$$
$$t \longmapsto (t,t^2)$$

$$\lambda_3 : \mathbb{G}_m \longrightarrow T$$
$$t \longmapsto (t,t)$$

- $V^{\lambda_0} = V$, $G^{\lambda_0} = G$, $G_{\lambda_0} = \{1\}$, $W_{\lambda_0} = W$, $k_{\lambda_0} = 1$,
  $\varepsilon_{V,\lambda_0} = \mathrm{triv}$

- $V^{\lambda_1} = (T^*(0 \oplus \mathbb{C}))^g$, $G^{\lambda_1} = T$, $G_{\lambda_1} = \mathbb{C}^* \times \{1\}$, $W_{\lambda_1} = \{1\}$,
  $\varepsilon_{V,\lambda_1} = \mathrm{triv}$ $\qquad k_{\lambda_1} = \dfrac{x_1{}^g}{x_1 - x_2}$

- $V^{\lambda_2} = \{0\}$, $G^{\lambda_2} = T$, $G_{\lambda_2} = T$, $W_{\lambda_2} = W$
  $k_{\lambda_2} = \dfrac{(x_1x_2)^g}{x_1 - x_2} \qquad \varepsilon_{V,\lambda_2} = \mathrm{sgn}$

• $V^{\lambda_3} = \{0\}$, $G^{\lambda_3} = G$, $G_{\lambda_3} = G$, $W_{\lambda_3} = W$,

$k_{\lambda_3} = (x_1 x_2)^g$, $\varepsilon_{V,\lambda_3} = \mathrm{sgn}$.

Some calculations:

$$P_{\lambda_0} = \bigoplus_{j=0}^{g-2} \mathbb{Q}(x_1+x_2)^j \subset H^*(V/G) \cong \mathbb{Q}[x_1+x_2, x_1x_2]$$

$$P_{\lambda_1} = \bigoplus_{j=0}^{g-1} \mathbb{Q}x_2^j \subset H^*(V^{\lambda_1}/G^{\lambda_1}) \cong \mathbb{Q}[x_1, x_1]$$

$$P_{\lambda_2} = \mathbb{Q} \subset H^*(V^{\lambda_2}/G^{\lambda_2}) \cong \mathbb{Q}[x_1, x_2]$$

$$P_{\lambda_3} = \{0\} \subset \mathbb{Q}[x_1+x_2, x_1x_2].$$

$$\mathrm{Ind}_{\lambda_1} : \mathbb{Q}[x_1, x_2] \longrightarrow \mathbb{Q}[x_1+x_2, x_1x_2]$$
$$f(x_1, x_2) \longmapsto \frac{x_1^g f(x_1,x_2) - x_2^g f(x_2,x_1)}{x_1 - x_2}$$

$$\mathrm{Ind}_{\lambda_2,\lambda_3} : \mathbb{Q}[x_1, x_2] \longrightarrow \mathbb{Q}[x_1+x_2, x_1x_2]$$
$$f(x_1, x_2) \longmapsto \frac{f(x_1,x_2) - f(x_2,x_1)}{x_1 - x_2}$$

surjective $\Longrightarrow P_{\lambda_3} = \{0\}$.

Integrality isomorphism

$$P_{\lambda_0} \oplus \left(P_{\lambda_1} \otimes \mathbb{Q}[x_1]\right) \oplus \left(P_{\lambda_2} \otimes \mathbb{Q}[x_1, x_2]\right)^{sgn} \longrightarrow \mathbb{Q}[x_1+x_2, x_1x_2]$$

$$(f, h, k) \mapsto f + \frac{x_1^g h(x_1, x_2) - x_2^g h(x_2, x_1)}{x_1 - x_2} + 2(x_1 x_2)^g \frac{k(x_1, x_2)}{x_1 - x_2}.$$

<u>exercise</u>: Check by hand this is an iso.

② $\mathbb{C}^* \curvearrowright V = \bigoplus_{k \in \mathbb{Z}} V_k$

For simplicity, we assume $V_0 = 0$.

$$\lambda_0 : \mathbb{C}^* \longrightarrow \mathbb{C}^*$$
$$t \longmapsto 1$$

$$\lambda_1 : \mathbb{C}^* \longrightarrow \mathbb{C}^*$$
$$t \longmapsto t$$

$\mathcal{P}_V = \{\overline{\lambda_0}, \overline{\lambda_1}\}$; no Weyl group

$V^{\lambda_0} = V$, $G^{\lambda_0} = G$, $G_{\lambda_0} = \{1\}$, $k_{\lambda_0} = 1$

$V^{\lambda_1} = pt$, $G^{\lambda_1} = G$, $G_{\lambda_1} = G$, $k_{\lambda_1} = \prod_{k>0} (kx)^{\dim V_k} \in \mathbb{Q}[x]$.

$$\mathrm{Ind}_{\lambda_1, \lambda_0} : \mathbb{Q}[x] \longrightarrow \mathbb{Q}[x]$$
$$f(x) \longmapsto k_{\lambda_1} \cdot f(x)$$
$$\qquad\qquad\qquad \shortparallel$$
$$\qquad\qquad\qquad C_V \cdot x^{\sum_{k>0} \dim V_k}.$$

$$P_{\lambda_0} = \mathbb{Q}[x]_{\deg < \sum_{k>0} \dim V_k}$$

$$P_{\lambda_1} = \mathbb{Q}.$$

Integrality isomorphism

$$P_{d_0} \oplus \left(P_{d_m} \otimes \mathbb{Q}[x]\right) \longrightarrow \mathbb{Q}[x]$$
$$(f,g) \longmapsto f + k_{d_m} \cdot g.$$

clearly an isomorphism

(6) Strengthening of the integrality isomorphism

(a) Identifying P_d

$$X_*(T)^{st} = \left\{ d \in X_*(T) \;\middle|\; \bigcup G^d/G_d\text{-orbits} \underset{\text{open}}{\overset{\text{closed}}{\subset}} V^d \right\}$$

+ generic stabilizer of a closed orbit is finite.

Conjecture: $P_d = \begin{cases} IH(V^d /\!/ G^d) & \text{if } d \in X_*(T)^{st} \\ 0 & \text{otherwise} \end{cases}$

- When (G,V) comes from a symmetric quiver: Meinhardt-Reineke 2014
- $(G = \mathbb{C}^*, V)$ (H, 2024)
- open in general

(b) Sheafifying the integrality isomorphism

$\pi_\lambda : V^\lambda/G^\lambda \longrightarrow V^\lambda/\!/G^\lambda \qquad d_\lambda = \dim V^\lambda - \dim G^\lambda$

$$\begin{tikzcd}
 & V^{\lambda\geq 0}/G^{\lambda\geq 0} \arrow[dl, "q_\lambda"'] \arrow[dr, "p_\lambda"] & \\
V^\lambda/G^\lambda \arrow[d, "\pi_\lambda"'] \arrow[rr, phantom, "\circlearrowright"] & & V/G \arrow[d, "\pi"] \\
V^\lambda/\!/G^\lambda \arrow[rr, "\imath_\lambda"'] & & V/\!/G
\end{tikzcd}$$

$\mathrm{Ind}_\lambda : (\imath_\lambda)_*(\pi_\lambda)_*\mathbb{Q}_{V^\lambda/G^\lambda}[d_\lambda] \longrightarrow \pi_*\mathbb{Q}_{V/G}[d]$.

[sheafified induction].

Theorem (H, 2024)

$\exists$ W_λ-equivariant constructible complexes $\mathcal{P}_\lambda$ on $V^\lambda/\!/G^\lambda$ s.t.

$$\bigoplus_{\tilde\lambda \in \mathcal{P}_V/W} \left((\imath_\lambda)_*\mathcal{P}_\lambda \otimes H^*_{G_\lambda}(pt)\right)^{\varepsilon_{V,\lambda}} \xrightarrow{\bigoplus_{\tilde\lambda}\mathrm{Ind}_\lambda} \pi_*\mathbb{Q}_{V/G}[d]$$

is an iso. in $\mathcal{D}^+(V/\!/G)$.

Conjecture (strengthening of the sheafified version)

$$P_\lambda \cong \begin{cases} IC(V^\lambda /\!/ G^\lambda)[-\dim G_\lambda] & \text{if } \lambda \in X_*(T)^{st} \\ 0 & \text{otherwise.} \end{cases}$$

7 - Construction of the P_λ's [vector space version]

V symmetric representation of G

$\lambda \in X_*(T)$ cocharacter

$V^\lambda \circlearrowleft G^\lambda \supset G_\lambda$

$\overline{G^\lambda} = G^\lambda / G_\lambda$ acts on V^λ; induction formalism for (G^λ, V^λ) instead of (G, V) gives

$$\mathrm{Ind}_{\mu,\lambda} : H^*\left((V^\lambda)^\mu / (G^\lambda)^\mu\right) \longrightarrow H^*(V^\lambda / G^\lambda)$$

P_λ = direct sum complement in $H^*_{\overline{G^\lambda}}(V^\lambda) \subset H^*_{G^\lambda}(V^\lambda)$ of

$$\sum_{\substack{\mu \in X_*(T) \\ ((V^\lambda)^\mu, (G^\lambda)^\mu) \neq (V^\lambda, G^\lambda)}} \mathrm{im}(\mathrm{Ind}_{\mu,\lambda}) \qquad \text{(all non-trivial inductions)}$$

8 - Further steps

Symplectic stacks and singularities

Weak Moment maps

X smooth variety $/\mathbb{C}$

$G \circlearrowright X$ action

$\exists\ \xi : TX \simeq T^*X$, $\exists\ \Psi : \mathfrak{g} \times X \simeq \mathfrak{g} \times X$ G-equivariants

$\exists\ \mu : X \longrightarrow \mathfrak{g}^*$ weak moment map

$$\begin{tikzcd} \mathfrak{g} \times X \arrow[r, "d\mu(-)(\xi)"] \arrow[d, "\Psi"'] \arrow[dr, phantom, "\circlearrowright"] & T^*X \\ \mathfrak{g} \times X \arrow[r, "a"', "\text{inf. action}"'] & TX \arrow[u, "S|\xi"'] \end{tikzcd}$$

actual moment map : $\Psi = id$.

ξ given by symplectic form on X.

G preserves the symplectic form.

Theorem (Halpern-Leistner)

Let $\mathfrak{M}$ be a derived stack with a good moduli space $\pi : \mathfrak{M} \longrightarrow \mathcal{M}$ such that $\exists\ \mathbb{T}_{\mathfrak{M}} \cong \mathbb{L}_{\mathfrak{M}}$. Then

$\forall x \in \mathcal{M}$, $\exists\ X$ smooth affine variety with G_x-action such that

$TX \cong T^*X$, and a weak moment map $\mu : X \to \mathfrak{g}^*$ s.t.
G_x-equiv

$$\begin{tikzcd}
\left(\left[\mu^{-1}(0)/G_x \right], 0 \right) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & (\mathfrak{M}, x) \arrow[d, "\pi"] \\
\left(\mu^{-1}(0) /\!\!/ G_x, 0 \right) \arrow[r] & (\mathcal{M}, x)
\end{tikzcd}$$

$\rightsquigarrow$ weak moment maps give local models for derived stacks with self-dual cotangent bundle.

Conjecture (HL) / Theorem (H, Davison)

$\mathfrak{M}$ 1-Artin derived stack with proper good moduli space $\mathcal{M}$. We assume that $\mathbb{L}_{\mathfrak{M}} \cong \mathbb{T}_{\mathfrak{M}}$. Then, $H^{BM}(\mathfrak{M})$ carries a pure MHS

Further goals: understand $IH^*(\mu^{-1}(0) /\!\!/ G)$ more precisely.