Instantons moduli spaces

GRIFT — 17-10-2023

(1) G compact group $= U(r)$

$\tilde{M}(G,n) = \{$ (anti) self dual G-connection on P principal G-bundle over $S^4 = \mathbb{R}^4 \cup \{\infty\}$, iso $P_\infty \cong G \}$ [cf Lotte talk last week]

some kinds of connections

$n =$ 2nd Chern class.
instanton number

instanton = finite energy ASD connection on $\mathbb{R}^4$; Uhlenbeck removable singularity theorem: such a connection extends to S^4.

(2) $G^{\mathbb{C}} =$ complexification of $G = Gl_r(\mathbb{C})$

$\mathbb{P}^2 = \mathbb{C}^2 \cup l_\infty$ "complex compactification of $\mathbb{R}^4 \cong \mathbb{C}^2$"

$\mathcal{O}\tilde{M}(G,n) = \{$ hol $G^{\mathbb{C}}$ bdle on $\mathbb{P}^2$, trivialization at $l_\infty \}$

Donaldson: $\tilde{M}(G,n) \cong G\tilde{M}(G^{\mathbb{C}},n)$ homeomorphism. $[G = U(r),$
$G^{\mathbb{C}} = GL(r,\mathbb{C})]$

Proof: quiver realisations of ① ADHM
② Barth (monad presentation of v. bundles on $\mathbb{P}^2$)
+ Kempf-Ness theorem: connection between HK quotient & GIT quotient.
(Underlying: Atiyah-Ward transform, going through sheaves on $\mathbb{P}^3$)

$\mathbb{R}^4 \cong \mathbb{C}^2$ choice of a complex structure
[quaternionic vector space: more complex structures]

Namely, we have a hyperkähler moment map:

$$GL(n,\mathbb{C}),\ U(n) \circlearrowright T^*\left(\mathrm{End}(\mathbb{C}^n) \oplus \mathrm{Hom}(\mathbb{C}^r,\mathbb{C}^n)\right)$$

$$\cong \mathrm{End}(\mathbb{C}^n)^{\oplus 2} \oplus \mathrm{Hom}(\mathbb{C}^r,\mathbb{C}^n) \oplus \mathrm{Hom}(\mathbb{C}^n,\mathbb{C}^r) \xrightarrow[\mu]{} \mathfrak{gl}(n) \times \mathfrak{u}(n) \cong \mathfrak{u}(n)^{\times 3}$$

$$((B_1, B_2),\quad I,\quad J) \qquad \mu = \mu_{\mathbb{C}} \times \mu_{\mathbb{R}}.$$

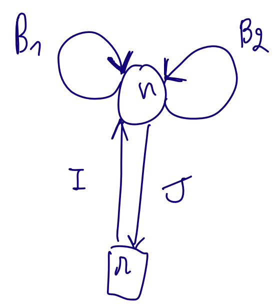
described as follows: $\mu_{\mathbb{C}}(B_1,B_2,I,J) = [B_1,B_2] + IJ$

$$\mu_{\mathbb{R}}(B_1,B_2,I,J) = \frac{\sqrt{-1}}{2}\left([B_1,B_1^\dagger] + [B_2,B_2^\dagger] + II^\dagger - J^\dagger J\right)$$

<u>ADHM</u>: $\tilde{M}(U(r),n) \underset{\text{open}}{\subseteq} \mu_{\mathbb{C}}^{-1}(0) \cap \mu_{\mathbb{R}}^{-1}(0) / U(n)$.
given by regular locus, i.e. where $U(n)$ acts freely

<u>Barth</u>: $G\tilde{M}(GL(r,\mathbb{C}),n) \underset{\text{open}}{\subseteq} \mu_{\mathbb{C}}^{-1}(0) /\!/ GL_n(\mathbb{C})$
$= \mathrm{Spec}\left(\mathbb{C}[\mu_{\mathbb{C}}^{-1}(0)]^{GL_n(\mathbb{C})}\right)$
given by regular locus, i.e. where $GL_n(\mathbb{C})$ acts freely.

<u>Kempf-Ness</u>: $\left(\mu_{\mathbb{C}}^{-1}(0) \cap \mu_{\mathbb{R}}^{-1}(0)\right)^{\mathrm{reg}} / U(n) \cong \mu_{\mathbb{C}}^{-1}(0)^{\mathrm{reg}} / GL_n(\mathbb{C})$ homeomorphism between the smooth opens. (regular)

<u>Quiver</u>: $Q=$ 

<u>Meaning</u>: in terms of vector spaces and group action.

<u>Rk</u>: Hyperkähler reduction.

$R(Q,r,n) := \mathrm{End}(\mathbb{C}^n)^{\oplus 2} \oplus \mathrm{Hom}(\mathbb{C}^r,\mathbb{C}^n) \oplus \mathrm{Hom}(\mathbb{C}^n,\mathbb{C}^r)$ is a Hyperkähler vector space: 3 complex structures $i, J, K = iJ$ which generate a sphere of $\mathbb{C}$-structures on $R(Q,r,n)$

- $\mathbb{C}^r, \mathbb{C}^n$ Hermitian in the standard way
$$(\underline{x},\underline{y}) = \sum_i x_i \overline{y_i}.$$
- $x \in \mathrm{Hom}(V,W) \rightsquigarrow x^* \in \mathrm{Hom}(W,V)$ adjoint
$$(xv,w) = (v, x^* w) \qquad v \in V, w \in W$$
- $R(Q,r,n)$ becomes Hermitian $\left((B_1,B_2,I,J),(B_1',B_2',I',J')\right)$
$= \mathrm{tr}(B_1 B_1'^*) + \mathrm{tr}(B_2 B_2'^*) + \mathrm{tr}(II'^*) + \mathrm{tr}(JJ'^*)$

- $J : R(Q,r,n) \longrightarrow R(Q,r,n)$ complex structure

  $(B_1, B_2, I, J) \mapsto (-B_2^*, B_1^*, J^*, -I^*)$
- i standard cpx structure
- $K = iJ$.

$\rightarrow R(Q,r,n)$ is $\left\{\begin{array}{l}\text{Hyperkähler vector space} \\ \text{quaternionic}\end{array}\right.$ i.e. $\mathbb{H}$-v.space.

$\rightarrow$ 3 symplectic forms : ω_i, $\omega_J + i\omega_K = \omega_{\mathbb{C}}$

$$\omega_i\left((B_1, B_2, I, J), (B_1', B_2', I', J')\right) = \mathrm{Im}\left(tr(B_1 B_1'^*) + tr(B_2 B_2'^*) + tr(I I'^*) + tr(J J'^*)\right)$$

$$\omega_{\mathbb{C}}\left(\qquad \right) = tr\left(B_2 B_1' - B_2' B_1 + I J' - I' J\right)$$

Action of $U(r) \curvearrowright R(Q,r,n)$ is tri-Hamiltonian i.e. Hamiltonian w.r.t. ω_i, ω_J and ω_K.

$\rightarrow$ 3 moment maps $\underset{\mu_{\mathbb{R}}}{\overset{\mu_i}{\shortparallel}}$, $\mu_{\mathbb{C}} = \mu_J + i\mu_K$

$\rightarrow \mu_i^{-1}(\xi_1) \cap \mu_{\mathbb{C}}^{-1}(\xi_2 + i\xi_3) / U(n)$ $\quad \xi_j \in \overset{\text{antihermitian matrices}}{\overset{\shortparallel}{\mathrm{Lie}\ U(n)}}$

is Hyperkähler quotient $\rightsquigarrow$ produces a new hyperkähler variety.

dimension $= \dim R(Q,r,n) - 4 \dim U(n)$.

Rk : HK structure is very rich; HK varieties are rather rare and hard to construct.

Partial compactification

For ② : vector bundles $\rightsquigarrow$ torsion free sheaves
|| Nakajima
locally free

we get a nicer space, partial compactification of the previous one

[corresponds to Uhlenbeck partial compactification on the ASD connections side; ideal instantons

$M(r,n)$ moduli space of framed rk r torsion-free sheaf on $\mathbb{P}^2$ w/ $c_2 = n$:

- $\mathcal{F}$ coherent sheaf on $\mathbb{P}^2$
- $\mathcal{F}$ is torsion free: any subsheaf supported on a closed subscheme is trivial
- $c_2(\mathcal{F}) = n$
- $\mathcal{F}|_{l_\infty} \cong \mathcal{O}_{l_\infty}^{\oplus r}$ isomorphism. $\quad l_\infty = \{[x:y:0]\}$

Nakajima: $M(r,n) \cong \mu^{-1}(0) /\!/_{\chi} GL_n$ where

$$= \mathrm{Proj}\left(\bigoplus_{m \geq 0} \mathbb{C}[\mu^{-1}(0)]^{\chi^m} \right). \quad \text{GIT.}$$

$\det^{-1} = \chi : GL_n \longrightarrow \mathbb{C}^*$ gives a linearization of the trivial line bundle on $\mu^{-1}(0)$.

= algebro-geometric model for the moduli space of instantons.

[Motivation for the definition of quiver varieties]

Rk By Kempf-Ness, One can realise $M(r,n)$ has HK reduction $\mu_{\mathbb{R}}^{-1}(i\mathrm{Id}) \cap \mu_{\mathbb{C}}^{-1}(0) / U(n)$.

$\boxed{\text{Last}}$: Hilbert scheme description in rk 1

$$M(1,n) \cong \mathrm{Hilb}^n(\mathbb{C}^2) = \{ \mathcal{J} \subset \mathcal{O}_{\mathbb{C}^2} \text{ codimension } n \text{ ideal} \}$$

Hilbert scheme of n points on $\mathbb{C}^2$

Proof: Map $\mathrm{Hilb}^n(\mathbb{C}^2) \longrightarrow M(1,n)$

$\mathcal{J}_Z \in \mathrm{Hilb}^n(\mathbb{C}^2)$

$Z \subset \mathbb{A}^2 \subset \mathbb{P}^2 \rightsquigarrow \tilde{\mathcal{J}}_Z$ ideal sheaf on $\mathbb{P}^2$ $\subset \mathcal{O}_{\mathbb{P}^2}$

$\Rightarrow \tilde{\mathcal{J}}_Z|_{\ell_\infty} \cong \mathcal{O}_{\ell_\infty}$

and $\tilde{\mathcal{J}}_Z$ torsion-free since $\mathcal{O}_{\mathbb{P}^2}$ torsion-free

rk 1 $\quad 0 \to \tilde{\mathcal{J}}_Z \to \mathcal{O}_{\mathbb{P}^2} \longrightarrow \mathcal{O}_{\mathbb{P}^2}/\tilde{\mathcal{J}}_Z \longrightarrow 0$

supported at fin many points in $\mathbb{P}^2$

$\Rightarrow$ rank 0.

Map $M(1,n) \longrightarrow \mathrm{Hilb}^n(\mathbb{C}^2)$

$$H^*(\mathbb{P}^2) \cong \mathbb{C}\cdot 1 \oplus \mathbb{C}\cdot[\ell_\infty] \oplus \mathbb{C}\cdot[\ell_\infty]^2.$$

$\mathcal{F}$ t.f. rk 1, $ch_2(\mathcal{F}) = n$

$$\mathcal{F}|_{\ell_\infty} \cong \mathcal{O}_{\ell_\infty} \Rightarrow ch_1(\mathcal{F})\cdot[\ell_\infty] = ch_1(\mathcal{F}|_{\ell_\infty}) = 0$$

$$\Rightarrow ch_1(\mathcal{F}) = 0$$

① $\mathcal{F} \underset{\text{iso on } \ell_\infty}{\hookrightarrow} \mathcal{F}^{**}$ bidual. $\mathcal{F}^{**}$ locally free $\Rightarrow$ line bdle

and $c_1(\mathcal{F}^{**}) = c_1(\mathcal{F}) = 0$

$$\mathcal{F}^{**} \cong \mathcal{O}_{\mathbb{P}^2}(d) \text{ for some } d$$

$\rightsquigarrow d = 0$

① $\mathcal{F} \hookrightarrow \mathcal{O}_{\mathbb{P}^2}$ iso on ℓ_∞ : $\mathcal{F}$ is an ideal sheaf defines a subscheme $Z \subset \mathbb{P}^2$.

and

$$\mathrm{rk}\left(\mathcal{O}_{\mathbb{P}^2}/\mathcal{F}\right) = 0$$

$$c_2\left(\mathcal{O}_{\mathbb{P}^2}/\mathcal{F}\right) = -n \Rightarrow \text{length } Z = n. \checkmark$$

<u>Proposition</u> (Poincaré polynomials)

$$\sum_{\substack{n=0 \\ \ell}}^{\infty} q^n t^\ell \dim H^\ell(\mathrm{Hilb}^n \mathbb{C}^2) = \prod_{m=1}^{\infty} \frac{1}{1-t^{2m-2}q^m} \quad \text{[Göttsche]}$$

$$Exp\left(\sum_{m=1}^{\infty} t^{2m-2} q^m\right)$$

$$= PE\left(\frac{1}{t^2} \frac{1}{1-qt^2}\right)$$

[to be compared later w/ the character of the Fock space]

AGT → correspondence
AGT → conjecture.

Physics : Correspondence between 2 physical theories

$U(r)$ gauge theory on $\mathbb{A}^2_{\mathbb{C}}$ $\mathbb{C}^\times \times \mathbb{C}^\times$- equivariant

Toda field theory of type A_{r-1}

(a CFT, generalisation of Liouville FT)

same partition function

Maths : Dictionary

Algebraic geometry ⟷ Representation theory

$$H^*_T(\underbrace{M(r,n)}_{\text{algebro-geometric incarnation of the mod space of instantons}})$$

Verma module for $W(gl_r)$
[symmetry algebra of the CFT]

$\cup$
1 unit cohomology class ⟷ Whittaker vector
"fundamental matter"

+ maybe other compatibilities.

AGT Theorem (SV, MO, BFN). There exists a natural representation

$$\underbrace{\mathcal{U}(W^{k}_{f_{pri}}(\mathfrak{gl}_r))}_{\text{Vertex algebra}} \longrightarrow \mathrm{End}\,(\mathbb{V}^{\circ}_{r}[\mathbb{C}^2])$$

[principal affine W-algebra; quantum Ham reduction of $V_k(\mathfrak{gl}_r)$]

algebra of modes.

such that $\mathbb{V}^{\circ}_{r}[\mathbb{C}^2]$ is identified w/ the universal Verma module M_r for $W^{k}_{f_{pri}}(\mathfrak{gl}_r)$

$$\mathbb{V}^{\circ}_{r}[\mathbb{C}^2] = \bigoplus_{n\in\mathbb{N}} H^{A}_{*}\big(\underbrace{M(r,n)}_{\text{instantons}}\big) \otimes_{H^{*}_{A}(pt)} \mathrm{Frac}\,(H^{*}_{A}(pt))$$

$$\uparrow U(r)$$

In English: Action of the affine W-algebra of G on the cohomology of the moduli space of G^{L}-instantons on $\mathbb{R}^4$.

One strategy

M_n geom spaces

$$H = \bigoplus_{n\in\mathbb{N}} H_*(M_n)$$

Want to construct an action of algebra A on H.

Convolution diagram

$$\begin{tikzcd} & M_{n,n+r} \arrow[d] & \\ & M_n \times M_{n+r} \arrow[dl] \arrow[dr] & \\ M_n & & M_{n+r} \end{tikzcd}$$

$M_{n,n+r} \leftarrow$ to be defined

$\rightsquigarrow$ operators $H_n \longrightarrow H_{n+r}$.

If A has a nice presentation (gen. & rels), can work.

AGT in rank 1

Focus : good for $r = 1$

Action of $\mathcal{H}$ Heisenberg Lie algebra on

$$\bigoplus_{n \in \mathbb{N}} H_*\left(M(1,n)\right) = \bigoplus_{n \in \mathbb{N}} H_*\left(\mathrm{Hilb}^n(\mathbb{C}^2)\right)$$

identifying it w/ the Fock space.

$$\mathcal{H} = \langle b_n, c, n \in \mathbb{Z} \setminus 0 \rangle \qquad \text{Heisenberg Lie algebra}$$

$$[b_n, b_m] = n\,\delta_{m+n}\,c .$$

c central

Fock space : $a \in \mathbb{C}^*$.

$\mathcal{F} = \mathbb{C}[p_1, p_2, \dots]$ polynomials in variables $p_1, p_2, \dots$

$\mathcal{H} \curvearrowright \mathcal{F}$ by

$$\begin{cases} b_m \longmapsto a m \dfrac{\partial}{\partial p_m} & m > 0 \\ p_{-m} \cdot & m < 0 \\ K \longmapsto a\,\mathrm{id}_{\mathcal{F}} \end{cases}$$

$$\mathcal{H} = \underbrace{\mathbb{C}\langle b_n : n < 0 \rangle}_{= \mathcal{H}_-} \oplus \mathbb{C} \cdot c \oplus \underbrace{\mathbb{C}\langle b_n : n > 0 \rangle}_{= \mathcal{H}_+}$$

$\mathcal{H}_+$ Abelian Lie algebra.

$\mathbb{C}c \oplus \mathcal{H}_+ \curvearrowright \mathbb{C}$ $\quad$ $c \mapsto$ mult by a

$\mathcal{H}_+ \mapsto$ trivial.

$\mathcal{F} \cong \mathcal{H} \otimes_{\mathbb{C}c \oplus \mathcal{H}_+} \mathbb{C}$: $\mathcal{F}$ is a highest weight representation of $\mathcal{H}$.

$$d_0 = \sum_m m p_m \frac{\partial}{\partial p_m} \curvearrowright \mathcal{F}$$

Character formula: $\operatorname{tr}_{\mathcal{F}} q^{d_0} := \sum_i q^i \dim \{v \in \mathcal{F} \mid d_0 v = i v\}$

$$= \prod_{j \geq 1} \frac{1}{1-q^j}$$

$$= \sum_{n \geq 1} q^n \dim H^*(\mathrm{Hilb}^n \mathbb{C}^2).$$

<u>Next step</u>: Make $\bigoplus_{n \geq 0} H^*(\mathrm{Hilb}^n \mathbb{C}^2)$ the Fock space.

$\rightsquigarrow$ Nakajima operators.

$X^{[n]} := \mathrm{Hilb}^n(\mathbb{C}^2)$

Nakajima operators : induced by correspondences and the convolution formalism.

$n \geq 0, \ \ell \geq 1$.
$$X^{[n+\ell, n]} \subset X^{[n+\ell]} \times X^{[n]}$$
$$\| $$
$$\{ (I \subset J) \ , \ \mathrm{supp}(J/I) \text{ is a point} \subset \mathbb{C}^2 \}$$

$$\begin{aligned} X^{[n+\ell, n]} &\longrightarrow \mathbb{C}^2 \\ (I \subset J) &\longmapsto \mathrm{supp}(J/I). \end{aligned}$$

$$\begin{tikzcd} & X^{[n+\ell,n]} \arrow[d, hook] & \\ & X^{[n+\ell]} \times \mathbb{C}^2 \times X^{[n]} \arrow[dl, "p_1"'] \arrow[d, "p_2"] \arrow[dr, "p_3"] & \\ X^{[n+\ell]} & \mathbb{C}^2 & X^{[n]} \end{tikzcd}$$

$$\begin{aligned} b_\ell : H_*(X^{[n]}) &\longrightarrow H_*(X^{[n+\ell]}) \\ \alpha &\longmapsto p_{1*}\left(p_{23}^*([\mathbb{C}^2] \otimes \alpha) \cap [X^{[n+\ell, n]}]\right) \end{aligned}$$

creation operator: adds points

$$\begin{aligned} b_{-\ell} : H_*(X^{[n+\ell]}) &\longrightarrow H_*(X^{[n]}) \\ \alpha &\longmapsto (-1)^\ell \, p_{3*}\left(p_{12}^*([pt] \otimes \alpha) \cap [X^{[n+\ell, n]}]\right) \end{aligned}$$

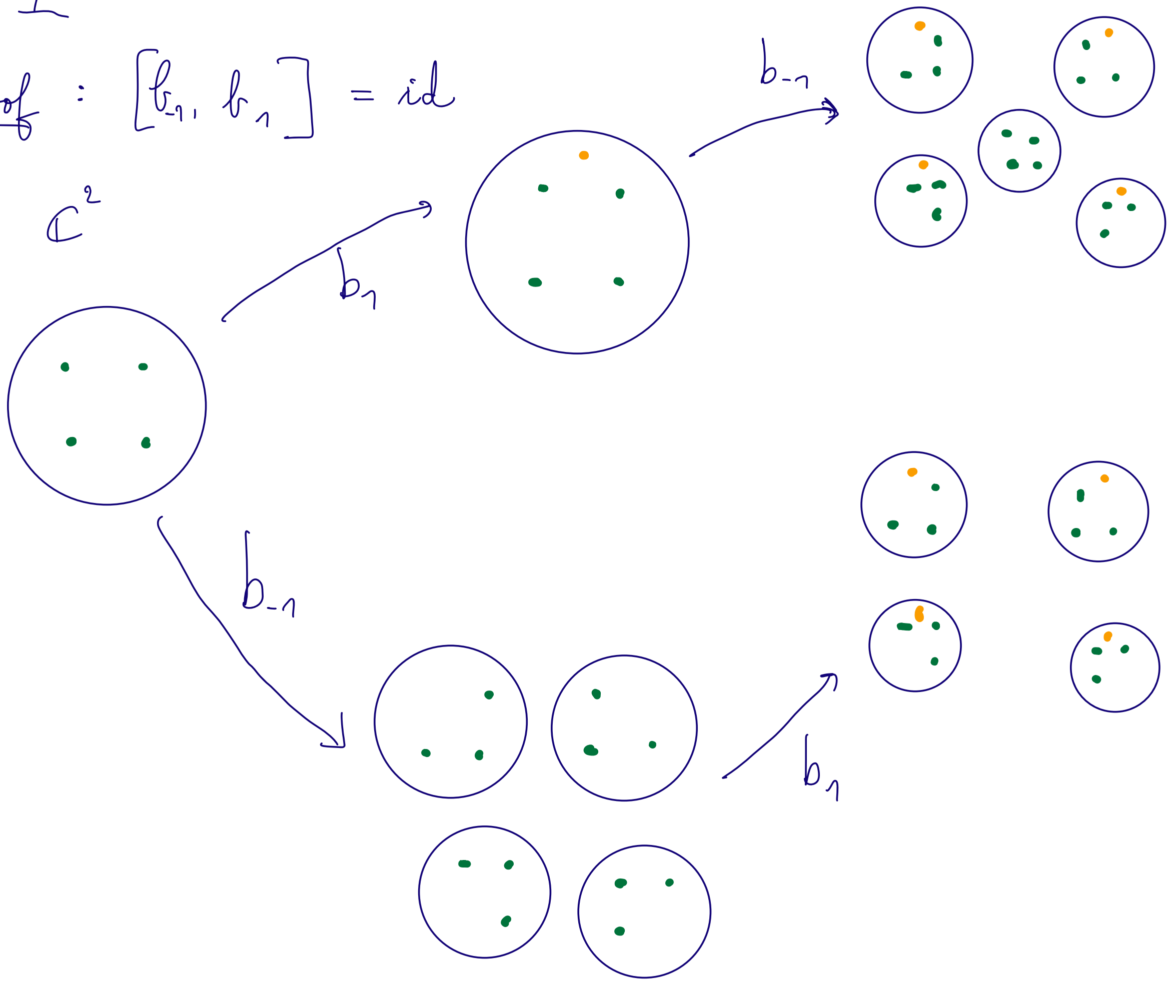
annihilation operator: delete points.

Theorem (Nakajima–Grojnowski)

$$[b_\ell, b_m] = \ell\,\delta_{\ell+m}\,\mathrm{id} \quad \text{on } \bigoplus_{n\geq 0} H_*(X^{[n]}).$$

Fock space representation of $\mathcal{H}$.

Proof: $[b_{-1}, b_1] = \mathrm{id}$



Details: intersection theory considerations.

Remark: We did not use the $\mathbb{C}^*$-action on $\mathbb{C}^2$. This story adapts well to equivariant cohomology and has been worked out by Vasserot

Higher ranks : Action of $W_{Lie}(\mathfrak{gl}(r))$ on some Lie algebra

$$H_r := \bigoplus_{n\geq 0} H^*(\underbrace{M(r,n)}_{\text{quiver variety interpretation.}}) .$$

possibly a torus.

Uses localisation techniques and stable envelopes theory.

* Caveat to adapt the rank one 1 method : no known presentation of $W_{Lie}(\mathfrak{gl}(r))$ by generators and relations.

* Solution : use auxiliary algebras, constructed geometrically and acting on H_r.

  * Yangian defined w/ RTT formalism and geometric R-matrices
  * W algebra implicitly described as kernel of screening operators.

$T = (\mathbb{C}^*)^r \curvearrowright M(r,n)$ by acting on the framing.

$$M(r,n)^T \cong M(1,n)^{\times r}$$

After localising,

$$\bigoplus_{n\geq 0} H^*(M(r,n)) \cong \bigoplus \underbrace{H^*(M(r,n)^T)}_{H^*(M(1,n))^{\otimes r}}$$

action of $\mathcal{H}^{\otimes r}$ r copies of Heisenberg.

Action of Heisenberg on $\bigoplus_{n \geq 0} H^*(M(1,n))$ obtained as in Nakajima or via Stable envelopes and a Yangian.

Then, use that $\mathcal{W}(\mathfrak{gl}(r))$ is the intersection of the kernels of screening operators (Feigin – Frenkel).
That is a long story.

Stable envelopes : long and complicated story!