II) Tools to study cohomological Hall algebras

Vanishing cycle sheaf functor : definition and first properties

→ one of the most important object in Donaldson–Thomas theory.

X complex algebraic variety

$f : X \longrightarrow \mathbb{A}^1$ regular function

$\mathbb{A}^1_{\leq 0} := \{ z \in \mathbb{C} \mid \mathrm{Re}(z) \leq 0 \}$

$X_0 = f^{-1}(0)$

$X_{\leq 0} = f^{-1}(\mathbb{A}^1_{\leq 0})$

$X_{>0} = X \setminus X_{\leq 0}$

nearby cycle functor : $\psi_f, \varphi_f : D^b(X) \longrightarrow D^b(X_0)$ ← a priori, categories of all sheaves of $\mathbb{Q}$-vspaces, not necessarily constructible ones.

$$\psi_f = (X_0 \hookrightarrow X)^* (X_{>0} \hookrightarrow X)_* (X_{>0} \hookrightarrow X)^* [-1]$$

$$\varphi_f = (X_0 \hookrightarrow X_{\leq 0})^* (X_{\leq 0} \hookrightarrow X)^!$$

distinguished triangle $\quad \varphi_f \rightarrow (X_0 \hookrightarrow X)^* \rightarrow \psi_f[1] \longrightarrow \quad$ functorial.

properties

* φ_f, ψ_f preserve constructible complexes [non obvious since constructed using non-algebraic maps]
* commute w/ Verdier duality
* functorialities w.r.t. proper morphisms
  $g : Y \to X$
  $g_0 : Y_0 \to X_0$.

  $\exists \quad \varphi_f \circ g_* \to g_{0*} \circ \varphi_{f \circ g}$, iso if g is proper

  $\exists \quad g_0^* \circ \varphi_f \to \varphi_{f \circ g} \circ g^*$, iso if g is smooth.
* If X is smooth, supp $\varphi_f(\mathbb{Q}_X) \subset \mathrm{crit}(f)$
* Thom–Sebastiani : $f : X \to \mathbb{A}^1$; $f' : X' \to \mathbb{A}^1 \rightsquigarrow f \boxplus f' : X \times X' \to \mathbb{A}^1$
  $(x, x') \mapsto f(x) + f(x')$.

$$\varphi_f(\mathcal{F}) \boxtimes \varphi_{f'}(\mathcal{G}) \cong \varphi_{f \boxplus f'}(\mathcal{F} \boxtimes \mathcal{G})\big|_{X_0 \times X_0'}.$$

$$f=0 \Rightarrow \varphi_f \simeq \mathrm{id}.$$

① Dimensional reduction

[Kontsevich-Soibelman, Davison, Kinjo
Deformed version: Davison-Padurariu

X smooth algebraic variety

$$\begin{tikzcd} E \arrow[d, "\pi"'] \\ X \end{tikzcd}$$ vector bundle

$$\begin{tikzcd} E^\vee \arrow[d, "\pi^\vee"] \\ X \arrow[u, bend left, "s"] \end{tikzcd}$$ dual vector bundle with section s.

$E \xrightarrow{f} \mathbb{A}^1$ regular function given by

$$E \simeq E \times_X X \xrightarrow[\mathrm{id}\times s]{} E \times_X E^\vee \xrightarrow{ev} \mathbb{A}^1$$

Define:

$$Z = s^{-1}(0) \subset X$$
$$\bar{Z} = \pi^{-1}(Z) \subset E$$
$$E_0 = f^{-1}(0) \overset{\imath}{\subset} E$$

Note that $\bar{Z} \subset_{\imath'} E_0$

$$\bar{\imath} = \imath \circ \imath' : \bar{Z} \to E$$

Then: $\pi_! \varphi_f(\mathrm{id} \to \bar{\imath}_* \bar{\imath}^*)\pi^*$ is an isomorphism of functors

Verdier duality: $\pi_* \varphi_f(\underbrace{\bar{\imath}_!}_{\bar{\imath}_*} \bar{\imath}^! \to \mathrm{id})\pi^!$ is an iso of functors

Because $f|_{\bar{Z}} = 0$, $\varphi_f\, \bar{\imath}_! \bar{\imath}^! = \bar{\imath}_! \bar{\imath}^!$ and so

$$\pi_* \bar{i}_* \bar{i}^! \simeq \pi_* \varphi_f \pi^!$$

Apply this to $\mathbb{D}\mathbb{Q}_X \cong \mathbb{Q}_X[2\dim X]$

$$\pi_* \bar{i}_* \bar{i}^! \mathbb{D}\mathbb{Q}_X \cong \pi_* \mathbb{D}\mathbb{Q}_{\bar{Z}}$$

$$\cong \mathbb{D}\mathbb{Q}_Z[2 \operatorname{rank} E]$$

and

$$\pi_* \varphi_f \pi^! \mathbb{D}\mathbb{Q}_X \cong \pi_* \varphi_f \mathbb{D}\mathbb{Q}_E$$

$$\cong \pi_* \varphi_f \mathbb{Q}_E[2\dim X + 2\operatorname{rank} E]$$

② Quivers with potential

Q quiver

$W \in \mathbb{C}[Q]$ linear combination of cyclic path

$=$ "potential"

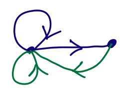
partial derivatives : $e \in Q_1$ arrow

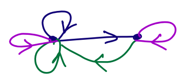
$$\frac{\partial}{\partial e}(a_1 \cdots a_r) = \sum_{a_i = e} a_{i+1} \cdots a_r a_1 \cdots a_{i-1}$$

linearly extended to any cyclic word.

Jacobi algebra: $\operatorname{Jac}(Q,W) := \mathbb{C}[Q] \Big/ \left\langle \frac{\partial W}{\partial e} : e \in Q_1 \right\rangle$

Important example $Q = (Q_0, Q_1)$ quiver 

$\overline{Q} = (Q_0, \overline{Q_1})$ double quiver 

$\tilde{Q} = (Q_0, \tilde{Q}_1)$ triple quiver 

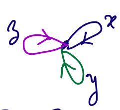
$$\tilde{Q}_1 = \overline{Q_1} \sqcup \{ \underbrace{\omega_i}_{\text{loop at the } i\text{-th vertex}} : i \in Q_0 \}$$

cubic potential for the triple quiver:

$$W = \left(\sum_{\alpha \in Q_1} [\alpha, \alpha^*] \right) \left(\sum_{i \in Q_0} \omega_i \right)$$

example: $Q =$  Jordan quiver

$\overline{Q} =$ 

$\tilde{Q} =$ 

$$W = [x,y]z = xyz - yxz$$

$$\frac{\partial W}{\partial x} = yz - zy \ ; \ \frac{\partial W}{\partial y} = zx - xz \ ; \ \frac{\partial W}{\partial z} = xy - yx$$

$$\mathrm{Jac}(\tilde{Q}, W) = \mathbb{C}\langle x,y,z\rangle / \text{commutators}$$

$$\cong \mathbb{C}[x,y,z]$$

In general, we have the following exercise

$$\mathrm{Jac}(\tilde{Q}, W) \cong \Pi_Q[\omega] \;.$$

Dimensional reduction: $H^*(\mathfrak{M}_{\tilde{Q}}, \varphi_{\mathrm{Tr}\,W} \mathbb{Q}_{\mathfrak{M}_{\tilde{Q}}}) \cong H^{BM}_*(\mathfrak{M}_{\Pi_Q}, \mathbb{Q}^{vir}_{\mathfrak{M}_{\Pi_Q}})$

More precisely, $\mathfrak{M}_{\tilde{Q},d} \xrightarrow{\pi} \mathfrak{M}_{\bar{Q},d}$ morphism forgetting the loops. Then,

$$\pi_* \varphi_{\mathrm{Tr}\,W} \mathbb{Q}^{vir}_{\mathfrak{M}_{\tilde{Q}}} \cong \mathbb{D}\mathbb{Q}^{vir}_{\mathfrak{M}_{\Pi_Q,d}}$$

Consequence: study $\mathfrak{M}_{\Pi_Q}$ by using $(\tilde{Q}, W)$.

Upshot:
* the most powerful constructions take place at the level of $(\tilde{Q}, W)$ [3CY level]
* the calculations more manageable at the level of Π_Q. [2CY level]

③ Perverse t-structures and perverse filtration

X $\mathbb{C}$-algebraic variety

$\mathcal{D}_c(X)$ constructible derived category of X.

Natural t-structure : $\mathcal{D}_c^{\leq 0}(X) = \{\mathcal{F} \in \mathcal{D}_c(X) \mid H^i(\mathcal{F}) = 0 \text{ for } i>0\}$

$\mathcal{D}^{\geq 0}(X) = \{\mathcal{F} \in \mathcal{D}_c(X) \mid H^i(\mathcal{F}) = 0 \text{ for } i<0\}$

Heart : $Sh_c(X) = \mathcal{D}_c^{\leq 0}(X) \cap \mathcal{D}_c^{\geq 0}(X)$ category of constructible sheaves over X.

More interesting : the perverse t-structure : [BBDG]

It has the perverse t-structure $({}^p\mathcal{D}_c^{\leq 0}(X), {}^p\mathcal{D}_c^{\geq 0}(X))$ whose heart $\mathrm{Perv}(X) = {}^p\mathcal{D}_c^{\leq 0}(X) \cap {}^p\mathcal{D}_c^{\geq 0}(X)$ is the abelian category of perverse sheaves.

${}^p\mathcal{D}_c^{\leq 0}(X)$ is the full subcategory of $\mathcal{D}_c(X)$ of complexes $\mathcal{F}$ such that :

① $\forall i \in \mathbb{Z}$, $\dim \{y \in Y \mid \mathcal{H}^i(\mathcal{F})|_y \neq 0\} \leq -i$ [support condition]

${}^p\mathcal{D}_c^{\geq 0}(X)$ is the full subcategory of $\mathcal{D}_c(X)$ of complexes $\mathcal{F}$ s.t.

② The complex $D\mathcal{F}$ satisfies ① [cosupport condition]

$\mathrm{Perv}(X) := {}^p\mathcal{D}_c^{\geq 0}(X) \cap {}^p\mathcal{D}_c^{\leq 0}(X)$ [support + cosupport].

Perverse truncations and perverse cohomologies

${}^p\tau^{\leq i}$, ${}^p\tau^{\geq i}$, ${}^p\mathcal{H}^i$

Semisimple complexes

$\mathcal{F} \in \mathcal{D}^+_c(X)$ is called semisimple if there exits an isomorphism

$$\mathcal{F} \cong \bigoplus_{i \in \mathbb{Z}} {}^p\mathcal{H}^i(\mathcal{F})[-i]$$

and each perverse sheaf ${}^p\mathcal{H}^i(\mathcal{F})$ is semisimple.

The main theorem in the theory of perverse sheaves is the following:

Decomposition theorem [BBDG]

Let $p: X \longrightarrow Y$ be a projective morphism between complex algebraic varieties and $\mathcal{F} = \mathcal{IC}(X)$. Then, $p_* \mathcal{F} \in \mathcal{D}_c(Y)$ is a semisimple complex.

Perverse filtration

$f: X \longrightarrow Y$ morphism between complex algebraic varieties

We obtain a filtration of the singular cohomology $H^*(X, \mathbb{Q})$ as follows:

$${}^p\tau_{\leq i}\, f_*\mathbb{Q}_X \longrightarrow f_*\mathbb{Q}_X \qquad \text{adjunction morphism}$$

$$F^i H^*(X, \mathbb{Q}) := \mathrm{Image}\left(H^*(Y, {}^p\tau_{\leq i} f_*\mathbb{Q}_X) \xrightarrow{\theta_i} H^*(Y, f_*\mathbb{Q}_X) \cong H^*(X, \mathbb{Q}) \right)$$

increasing filtration.

If $f_*\mathbb{Q}_X \in \mathcal{D}^+_c(Y)$ is semisimple, the filtration is split and the

↳ e.g. if X is smooth and f is proper.

θ_i are injective.

(4) The BPS Lie algebra and the BPS algebra

We define the BPS Lie algebra for $(\tilde{Q}, W)/\Pi_Q$

Recall $\mathfrak{M}_{\tilde{Q}} = \bigsqcup_{d \in \mathbb{N}^{Q_0}} \mathfrak{M}_{\tilde{Q},d}$; $\mathfrak{M}_{\tilde{Q},d} = X_{\tilde{Q},d}/GL_d$ is the stack of d-dimensional representations of $\tilde{Q}$.

$\mathcal{M}_{\tilde{Q}} = \bigsqcup_{d \in \mathbb{N}^{Q_0}} \mathcal{M}_{\tilde{Q},d}$; $\mathcal{M}_{\tilde{Q},d} = X_{\tilde{Q},d}/\!/GL_d = \operatorname{Spec} \mathbb{C}[X_{\tilde{Q},d}]^{GL_d}$ is the affine GIT quotient.

$JH : \mathfrak{M}_{\tilde{Q}} \longrightarrow \mathcal{M}_{\tilde{Q}}$ is the natural morphism.

Proposition: $JH_* \mathbb{Q}^{\mathrm{vir}}_{\mathfrak{M}_{\tilde{Q}}} \in \mathcal{D}^+_c(\mathcal{M}_{\tilde{Q}})$ is a semisimple complex

__Proof__: Approximation by proper maps of the morphism $\mathfrak{M}_{\tilde{Q}} \to \mathcal{M}_{\tilde{Q}}$ from the stack to the good moduli space.

Smallness

Let Q be a symmetric quiver. Then, there exists a locally closed stratification $(S_\xi)_{\xi \in \Xi}$ of $\mathcal{M}_Q$ s.t. $p_\xi : \mathcal{S}_\xi = p^{-1}(S_\xi) \to S_\xi$ satisfy:

(i) p_ξ is an étale locally trivial fibration

(ii) $\forall s_\xi \in S_\xi$,

$$\dim S_\xi + 2 \dim p^{-1}(s_\xi) \leq \dim V/G$$

with equality iff S_ξ is an open stratum and $\mathcal{S}_\xi$ has finite stabilizers.

Consequence: $JH_* \mathbb{Q}^{vir}_{\mathfrak{M}_{\tilde{Q}}} \in {}^p\mathcal{D}^{\geq 1}(\mathcal{M}_{\tilde{Q}})$. [exercise]

Apply $\varphi_{\mathrm{Tr}\,W}$: $JH_* \varphi_{\mathrm{Tr}\,W} \mathbb{Q}^{vir}_{\mathfrak{M}_{\tilde{Q}}} \in {}^p\mathcal{D}^{\geq 1}(\mathcal{M}_{\tilde{Q}})$ since vanishing cycle is perverse t-exact.

Define $\mathcal{BPS}_{(\tilde{Q},W)} := {}^p\mathcal{H}^1(JH_* \varphi_{\mathrm{Tr}\,W}) \in \mathrm{Perv}(\mathcal{M}_{\tilde{Q}})$ BPS Lie algebra sheaf.

Support lemma $\mathcal{M}_{\overline{Q}} \times \mathbb{A}^1 \overset{i}{\hookrightarrow} \mathcal{M}_{\tilde{Q}}$.

supp $\mathcal{BPS}_{(\tilde{Q},W)} \subset$ image of i

$+$ $\mathcal{BPS}_{(\tilde{Q},W)}$ is $\mathbb{A}^1$-equivariant.

$$\Rightarrow \mathcal{BPS}_{(\tilde{Q},W)} \simeq \underbrace{\mathcal{BPS}^{red}_{(\tilde{Q},W)}}_{\text{"dimensionally reduced BPS sheaf"}} \boxtimes \mathbb{Q}_{\mathbb{A}^1}[1].$$

$\mathcal{BPS}^{red}_{(\tilde{Q},W)}$ is supported on $\mathcal{M}_{\Pi_Q} \subset \mathcal{M}_{\overline{Q}}$

$\in \mathrm{Perv}(\mathcal{M}_{\Pi_Q})$.

⑤ The PBW theorem

Thm (Davison) There are isomorphisms in $\mathcal{D}_c^+(\mathcal{M}_{\tilde{Q}}, \mathbb{Q})$

$$JH_* \varphi_{\mathrm{Tr}\,W} \mathbb{Q}^{\mathrm{vir}}_{\mathfrak{M}_{\tilde{Q}}} \cong \mathrm{Sym}\left(\mathcal{BPS}_{(\tilde{Q},W)}[-1] \otimes H^*_{\mathbb{C}^*}(pt)\right)$$

$$JH_* \mathbb{DQ}^{\mathrm{vir}}_{\mathfrak{M}_{\Pi_Q}} \cong \mathrm{Sym}\left(\mathcal{BPS}^{\mathrm{red}}_{(\tilde{Q},W)} \otimes H^*_{\mathbb{C}^*}(pt)\right)$$

Consequences: The perverse filtration on $JH_* \mathbb{DQ}^{\mathrm{vir}}_{\mathfrak{M}_{\Pi_Q}}$ starts in degree 0.

Thm (Davison) $\underbrace{{}^p\mathcal{H}^0\left(JH_* \mathbb{DQ}^{\mathrm{vir}}_{\mathfrak{M}_{\Pi_Q}}\right)}_{= \mathcal{BPS}_{Alg}} \in \mathrm{Perv}(\mathcal{M}_{\Pi_Q})$ has an induced algebra structure.

$${}^p\mathcal{H}^0\left(JH_* \mathbb{DQ}^{\mathrm{vir}}_{\mathfrak{M}_{\Pi_Q}}\right) \underset{\text{algebras}}{\cong} \mathcal{U}\left(\mathcal{BPS}^{\mathrm{red}}_{(\tilde{Q},W)}\right)$$

Questions: Is it possible to describe the structure of the Lie algebra object $\mathcal{BPS}^{\mathrm{red}}_{(\tilde{Q},W)} \in \mathrm{Perv}(\mathcal{M}_{\Pi_Q})$ (and so obtain the structure of the Lie algebra $\mathrm{BPS}^{\mathrm{red}}_{(\tilde{Q},W)} := H^*(\mathcal{M}_{\Pi_Q}, \mathcal{BPS}_{(\tilde{Q},W)})$) ?

⟶ Next time: it has the structure of a generalized Kac-Moody Lie algebra.