Goal of the exercise and research sessions :

Explain the structure of $\mathrm{CoHA}(\Pi_Q)$

Q quiver

$\overline{Q}$ double quiver

$\Pi_Q = \mathbb{C}\overline{Q} \Big/ \sum_{\alpha \in Q_1} [\alpha, \alpha^*]$ ≈ 2-Calabi-Yau completion of $\mathbb{C}Q$.

$\mathfrak{M}_{\Pi_Q}$ = stack of representations of $\mathrm{Rep}\,\Pi_Q$

$\mathrm{CoHA}(\Pi_Q) = H_*^{BM}(\mathfrak{M}_{\Pi_Q}, \mathbb{Q})$ + associative algebra structure.

Thm [Davison-H-Schlegel Mejia]

$$\underset{\substack{\uparrow \\ \text{vector space}}}{\mathrm{CoHA}(\Pi_Q) \cong} \mathrm{Sym}\left(\mathfrak{n}^+_{BPS} \otimes H^*_{\mathbb{G}_m}(pt)\right)$$

$\underbrace{\mathfrak{n}^+_{BPS}}$ is the BPS Lie algebra

generalized Kac-Moody Lie algebra generated par

$IH^*(\mathcal{M}_{\Pi_Q, d})$, $d \in \Sigma_Q \subset \mathbb{N}^{Q_0}$.

Some exercises

* (Trivial) exercise:

  Determine $\mathcal{IC}(\mathbb{Q}_{\mathbb{A}^1})$, $\mathbb{D}\mathbb{Q}_{\mathbb{A}^1}$ $\in \mathcal{D}^b_c(\mathbb{A}^1, \mathbb{Q})$

* Compute $H^*(\{xy=0\}, \mathbb{Q})$

  $H_*^{BM}(\{xy=0\}, \mathbb{Q})$

  $IH^*(\{xy=0\}, \mathbb{Q})$

* Determine $\pi_* \mathbb{Q}_{\mathbb{A}^2/\mathbb{G}_m}$ where $\pi : \underbrace{\mathbb{A}^2/\mathbb{G}_m}_{\text{quotient stack}} \to \mathbb{A}^1$, $(x,y) \mapsto xy$

  and $\mathbb{G}_m$ acts on $\mathbb{A}^2$ w/ weight 1 on x, -1 on y.

* Show that $\mathbb{D}(\mathbb{Q}_{\{xy=0\}}[1])$ is perverse and find its Jordan-Hölder filtration. Use it to compute $H_*^{BM}(\{xy=0\}, \mathbb{Q})$.

* Show that $\sum_{i \geq 0} \dim H_i^{BM}(\{xy=0\}/\mathbb{C}^*, \mathbb{Q})\, q^{i/2}$

  $$= \frac{2-q}{1-q}$$

$\ast$ Very difficult: determine the generating series

$$\sum_{n\geq 0} \dim H_i^{BM}\left(\mathcal{E}(\mathfrak{gl}_n)/GL_n, \mathbb{Q}\right) q^{i/2} t^n$$
$$= \mathrm{Exp}\left(\frac{qt^{-1}}{1-t}\right)$$

plethystic exponential

where $\mathcal{E}(\mathfrak{gl}_n) = \{x, y \in \mathfrak{gl}_n \mid [x,y] = 0\}$.

$\ast$ Easy: Determine $\pi_* \mathbb{Q}_{pt/GL_n}[-n^2]$ where $\pi : pt/GL_n \to pt$.

$\ast$ Difficult: Determine $\pi_* \mathbb{Q}_{\mathfrak{gl}_n/GL_n} \in \mathcal{D}_c^+(\mathbb{C}^n/\mathfrak{S}_n, \mathbb{Q})$ where
$\pi : \mathfrak{gl}_n/GL_n \longrightarrow \mathfrak{gl}_n /\!/ GL_n \cong \mathbb{C}^n /\!/ \mathfrak{S}_n$.

$\ast$ Determine $\dim_{\mathbb{C}} \mathcal{E}(\mathfrak{gl}_n)$

$\ast$ Very difficult:
Determine $\pi_* \mathbb{DQ}_{\mathcal{E}(\mathfrak{gl}_n)/GL_n}$ where

$$\pi : \mathcal{E}(\mathfrak{gl}_n)/GL_n \to \mathcal{E}(\mathfrak{gl}_n)/\!/GL_n \cong (\mathbb{C}^2)^n /\!/ \mathfrak{S}_n$$

for $n = 1$, $n = 2$, $\underbrace{\text{general } n}_{\text{hard}}$.

* $Q = (Q_0, Q_1)$ quiver

Assume Q has no loops

$d \in \mathbb{N}^{Q_0}$

$$X_{Q,d} = \prod_{\alpha \in Q_1} \mathrm{Hom}(\mathbb{C}^{d_{s(\alpha)}}, \mathbb{C}^{d_{t(\alpha)}})$$

$$T^* X_{Q,d} \xrightarrow{\mu_d} \mathfrak{gl}_d \quad \text{moment map}$$

$$\Lambda_{Q,d} := \mu_d^{-1}(0) \cap \pi_d^{-1}(0) \quad \text{where}$$

$$\pi_d : T^* X_{Q,d} \longrightarrow T^* X_{Q,d} /\!\!/ GL_d \quad \text{quotient morphism.}$$

Show that $\Lambda_{Q,d}$ is a Lagrangian subvariety of $T^* X_{Q,d}$.

* X smooth G-variety.

Give the moment map for the G-action on $T^* X$.

* Take $Q = \bullet \xrightarrow{(r)} \bullet$ r arrows

Determine $\Lambda_{Q,(1,1)}$

Determine $\# \mathrm{Rep}_Q(\mathbb{F}_q)/\mathrm{iso}$.

(I) Examples of cohomological Hall algebras – The CoHA product

① Constructible derived category [6-operations, CoHA at the sheaf level]

X complex algebraic variety

$\mathcal{D}_c^b(X,\mathbb{Q})$ = bounded derived category of X of sheaves of $\mathbb{Q}$-vector spaces.

objects:

$$\cdots \to \mathcal{F}^{-1} \xrightarrow{d^{-1}} \mathcal{F}^0 \xrightarrow{d^0} \mathcal{F}^1 \longrightarrow \cdots \qquad d^i \circ d^{i-1} = 0.$$

morphisms are

$$\begin{tikzcd} & \mathcal{H}^\bullet \arrow[dl, "g"'] \arrow[dr, "h"] & \\ \mathcal{F}^\bullet & & \mathcal{G}^\bullet \end{tikzcd}$$

g, h are morphisms of complexes

g induces $H^i(\mathcal{H}^\bullet) \cong H^i(\mathcal{F}^\bullet)$ ["quasi-isomorphism"]

translation functor : $\mathcal{F}[1]^i = \mathcal{F}^{i+1}$; $d^i_{\mathcal{F}[1]} = -d^{i+1}_{\mathcal{F}}$

② operations

$f : X \longrightarrow Y$ morphism of complex algebraic varieties

$f^*, f^! : \mathcal{D}_c^b(Y) \longrightarrow \mathcal{D}_c^b(X)$

$f_*, f_! : \mathcal{D}_c^b(X) \longrightarrow \mathcal{D}_c^b(Y)$

functors are all derived.

$\mathbb{D}_X : \mathcal{D}_c^b(X)^{op} \longrightarrow \mathcal{D}_c^b(X)$ Verdier duality

$\mathbb{D}_X \circ \mathbb{D}_X \simeq \mathrm{id}$

$\otimes^{\mathbb{L}}$

$\mathcal{H}om(-,-)$

$\mathcal{H}om(\mathcal{F},\mathcal{G}) \cong \mathbb{D}\mathcal{F} \otimes^{\mathbb{L}} \mathcal{G}$

adjunctions : (f^*, f_*) ; $(f_!, f^!)$, $(\otimes^{\mathbb{L}}, \mathcal{H}om)$

$\mathbb{D}_X \mathcal{F} \cong \mathcal{H}om(\mathcal{F}, pt^! \mathbb{Q})$ where $pt : X \longrightarrow pt$ unique morphism.

③ Some useful properties

For X smooth, $\mathbb{D}\mathbb{Q}_X \cong \mathbb{Q}_X[2\dim X]$

For $f: X \to Y$ smooth, $f^! \cong f^*[2\dim f]$

If f is proper, $f_! \cong f_*$

If $X = pt$, $\mathcal{D}^b_c(pt) \cong$ $\mathbb{Z}$-graded vector spaces w/ fin dim cohomology

$$\mathcal{F}^\bullet \mapsto \bigoplus_{i\in\mathbb{Z}} H^i(\mathcal{F}^\bullet)[-i]$$

<u>Base-change</u>

$$\begin{tikzcd} X \arrow[r, "f'"] \arrow[d, "g'"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "g"] \\ Z \arrow[r, "f"'] & T \end{tikzcd}$$

$g^! f_* \cong f_! g'^!$ isomorphism of functors

④ Cohomologies

$\pi : X \to pt$

Singular cohomology : $H^*(X,\mathbb{Q}) \cong \pi_* \mathbb{Q}$

More generally, cohomology w/ coefficients : $\mathcal{F} \in \mathcal{D}^b_c(X,\mathbb{Q})$;

$$H^*(X,\mathcal{F}) := \pi_* \mathcal{F}.$$

Borel-Moore homology $\quad H^{BM}_i(X,\mathbb{Q}) \cong H^{-i}(X, \mathbb{D}\mathbb{Q}_X)$

Compactly supported cohomology $\quad H^*_c(X,\mathbb{Q}) \cong H^{BM}_*(X,\mathbb{D}\mathbb{Q}_X)^\vee$

intersection cohomology $\quad IH^*(X,\mathbb{Q}) \cong H^*(X, \mathcal{IC}(\mathbb{Q}_X))$

(5) Important distinguished triangles

$X \qquad j : U \hookrightarrow X$ open and $i : F \hookrightarrow X$ closed complement

(i) $j_! j^* \to \mathrm{id} \to i_* i^* \to$ functorial triangles

(ii) $i_* i^! \to \mathrm{id} \to j_* j^* \to$

(ii) applied to $\mathbb{D}\mathbb{Q}_X$ gives the long exact sequence in Borel-Moore homology

$$\cdots \to H^{BM}_{-i}(F) \to H^{BM}_{-i}(X) \to H^{BM}_{-i}(U) \to H^{BM}_{-(i+1)}(F) \to \cdots$$

The first triangle (i) is dual and gives the long exact sequence in compactly supported cohomology.

Computation of Borel-Moore homology

BM-homology tends to be much richer than just cohomology for the class of spaces we will be considering: usually contractible spaces but singular with several irreducible components.

<u>example</u> : $X = \{xy = 0\} \subset \mathbb{C}^2 \qquad \{0\} \subset X \supset \mathbb{C}^* \sqcup \mathbb{C}^*$

$$H^{BM}_{-i}(\{0\}) \to H^{BM}_{-i}(X) \to H^{BM}_{-i}(\mathbb{C}^*) \oplus H^{BM}_{-i}(\mathbb{C}^*) \to$$

$$H^i(\mathbb{C}^*, \mathbb{D}\mathbb{Q}) \simeq H^{i+2}(\mathbb{C}^*, \mathbb{Q})$$

$$\simeq \begin{cases} \mathbb{Q} & i = -2 \\ \mathbb{Q} & i = -1 \end{cases}$$

$i = -2$

$$\{0\} \longrightarrow H_2^{BM}(X) \longrightarrow \underbrace{H_2^{BM}(\mathbb{C}^*)^{\oplus 2}}_{\cong\, \mathbb{Q}^2} \longrightarrow H_1^{BM}(\{0\}) = 0$$

$$\{0\} \longrightarrow H_1^{BM}(X) \longrightarrow \underbrace{H_1^{BM}(\mathbb{C}^*)^{\oplus 2}}_{\cong\, \mathbb{Q}^2} \longrightarrow \underbrace{H_0^{BM}(\{0\})}_{\cong\, \mathbb{Q}} \to H_0^{BM}(X) \to 0$$

$$\mathbb{Q} \longrightarrow H_0^{BM}(X) \longrightarrow 0$$

$$\Rightarrow \quad H_2^{BM}(X) \cong \mathbb{Q}^{\oplus 2}$$

$H_1^{BM}(X) = \mathbb{Q}$, $\quad H_0^{BM}(X) = 0$ using $\{x=0\} \subset X \hookleftarrow \mathbb{C}^*$

or ~~$H_1^{BM}(X) = \mathbb{Q}^{\oplus 2}$, $H_0^{BM}(X) = \mathbb{Q}$~~

fundamental classes: if X is a complex algebraic variety, possibly reducible, of dim n, $\quad H_{2n}^{BM}(X, \mathbb{Q})$ has a basis indexed by n-dim irreducible components of X.

Equivariant constructible derived category

In geometric representation theory, one needs to take symmetries into account and so work with equivariant constructible derived categories instead.

$X \circlearrowleft G$ complex algebraic variety with G-action.

Bernstein–Lunts: $\mathcal{D}_{c,G}(X, \mathbb{Q})$

extend the functors previously defined to it. We ignore the subtleties.

In terms of stacks, $\mathcal{D}_{c,G}(X,\mathbb{Q})$ is the constructible derived category of the stack X/G.

Local complete intersection morphisms

l.c.i. morphisms are the ones for which one can do pullback in Borel–Moore homology.

$X \xrightarrow{f} Y$ is "l.c.i" if f can be factored
$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "g"'] & & Y \\ & Z \arrow[ur, "h"'] & \end{tikzcd}$$
with g regular closed immersion and h smooth.

<u>example</u>: If X, Y are smooth,
$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr] & & Y \\ & X \times Y \arrow[ur, "pr_2"'] & \end{tikzcd}$$
$x \mapsto (x, f(x))$

More generally, a section of a smooth morphism is l.c.i since regular closed immersion:
$$\begin{tikzcd} Y \arrow[r, "p"] & X \arrow[l, bend left, "f"] \end{tikzcd}$$
smooth $\quad p \circ f = id_X$

Pullback for l.c.i morphisms

$f : X \to Y$ with X, Y smooth. Define $\operatorname{codim} f := \dim Y - \dim X$
$= -$ rel. dim

$$\left.\begin{array}{l} D\mathbb{Q}_Y \cong \mathbb{Q}_Y[2\dim Y] \\ D\mathbb{Q}_X \cong \mathbb{Q}_X[2\dim X] \end{array}\right\} \Rightarrow f^* D\mathbb{Q}_Y \cong D\mathbb{Q}_X[2 \operatorname{codim} f].$$

By adjunction, $D\mathbb{Q}_Y \longrightarrow f_* D\mathbb{Q}_X[2 \operatorname{codim} f]$

By taking derived global sections,
$$H_*^{BM}(Y, \mathbb{Q}) \longrightarrow H^{BM}_{*-2\operatorname{codim} f}(X, \mathbb{Q}) :$$ pullback in BM. homology.

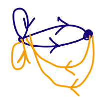
④ 2d Cohomological Hall algebras

$Q = (Q_0, Q_1)$ quiver

vertices arrows

$\overline{Q} = (Q_0, \overline{Q}_1)$ double quiver

$\overline{Q}_1 = Q_1 \sqcup Q_1^*$ s.t.

(Q_0, Q_1^*) is the opposite of Q : $\overline{Q} =$ 

representation space

$d \in \mathbb{N}^{Q_0}$ dimension vector

$$X_{Q,d} := \bigoplus_{\alpha \in Q_1} \mathrm{Hom}\left(\mathbb{C}^{d_{s(\alpha)}}, \mathbb{C}^{d_{t(\alpha)}}\right)$$

action of $GL_d := \prod_{i \in Q_0} GL_{d_i}$

$Q \rightsquigarrow \overline{Q}$: $X_{\overline{Q},d} \cong T^* X_{Q,d}$ trace pairing

$T^* X_{Q,d}$ is a symplectic vector space : natural symplectic form

$$\omega : X_{\overline{Q},d} \times X_{\overline{Q},d} \longrightarrow \mathbb{C}$$
$$\left((x_\alpha, x_{\alpha^*})_{\alpha \in Q_1}, (y_\alpha, y_{\alpha^*})_{\alpha \in Q_1}\right) \longmapsto \mathrm{Tr}\left(\sum_{\alpha \in Q_1} x_\alpha y_{\alpha^*} - x_{\alpha^*} y_\alpha\right)$$

The action of GL_d on $T^* X_{Q,d}$ is Hamiltonian, with moment map

$$\mu_d : X_{\overline{Q},d} \longrightarrow \mathfrak{gl}_d \cong \mathfrak{gl}_d^*$$
$$(x_\alpha, x_{\alpha^*}) \longmapsto \sum_{\alpha \in Q_1} [x_\alpha, x_{\alpha^*}]$$

For $d, e \in \mathbb{N}^{Q_0}$, $\mathbb{C}^{d+e} \cong \mathbb{C}^d \oplus \mathbb{C}^e$

$$X_{Q,d,e} = \{ x \in X_{Q,d+e} \mid x(\mathbb{C}^d) \subset \mathbb{C}^d \} = \begin{pmatrix} \boxed{*} \\ 0 \;\; \end{pmatrix}$$

$$\mathfrak{n}_{d,e} = \left\{ \begin{pmatrix} 0 & \boxed{*} \\ 0 & 0 \end{pmatrix} \right\} \subset \mathfrak{gl}_{d,e} = \left\{ \begin{array}{c} \\ d\left\{ \right. \\ \\ \end{array} \begin{pmatrix} * & * \\ & * \\ & \underbrace{}_{e} \end{pmatrix} \right\} \subset \mathfrak{gl}_{d+e}$$

$$GL_{d,e} = \left\{ \begin{pmatrix} * & * \\ & * \end{pmatrix} \right\} \subset GL_{d+e}$$

Euler forms : $d, e \in \mathbb{N}^{Q_0}$

$$\chi_Q(d,e) = \sum_{i \in Q_0} d_i e_i - \sum_{\alpha \in Q_1} d_{s(\alpha)} e_{t(\alpha)}$$

$$\chi_Q(d,d) = \dim GL_d - \dim X_{Q,d}$$

$$\chi_{\bar{Q},d}(d,e) = \sum_{i \in Q_0} d_i e_i - \sum_{\alpha \in Q_1} \left(d_{s(\alpha)} e_{t(\alpha)} + e_{s(\alpha)} d_{t(\alpha)} \right)$$

Remark: preprojective algebra $\Pi_Q = \mathbb{C}\bar{Q} \Big/ \sum_{\alpha \in Q_1} [\alpha, \alpha^*]$.

$\mu_d^{-1}(0)/GL_d$ is the stack of d-dim'l representations of Π_Q

$\mathcal{M}_{\Pi_Q,d} = \mu_d^{-1}(0) /\!\!/ GL_d$ moduli space ...

Induction diagram for 2d cohomological Hall algebras

$$
\begin{tikzcd}
\mathcal{M}_{\Pi_Q,d} \times \mathcal{M}_{\Pi_Q,e} \arrow[rr, "\oplus_{d,e}"] & \circlearrowright & \mathcal{M}_{\Pi_Q,d+e} \\
\mathrm{GL}_d \times \mathrm{GL}_e \circlearrowright \mu_d^{-1}(0) \times \mu_e^{-1}(0) \arrow[u, "\pi_d \times \pi_e"] \arrow[d, "i_d \times i_e"'] & \mathrm{GL}_{d,e} \circlearrowright \mu_{d,e}^{-1}(0) \arrow[l, "q'_{d,e}"'] \arrow[r, "p'_{d,e}"] \arrow[d, "i_{d,e}"] \arrow[dl, phantom, "\llcorner", very near start] & \mathrm{GL}_{d+e} \circlearrowright \mu_{d+e}^{-1}(0) \arrow[u, "\pi_{d+e}"'] \arrow[d, "i_{d+e}"] \\
X_{\bar{Q},d} \times X_{\bar{Q},e} \times \mathfrak{n}_{d,e} & X_{\bar{Q},d,e} \arrow[l, "q_{d,e}"'] \arrow[r, "p_{d,e}"] & X_{\bar{Q},d+e}
\end{tikzcd}
$$

$$\left(\boxed{*} \quad \boxed{*} \quad \boxed{*} \right) \longleftarrow\!\!\shortmid \; x = \begin{pmatrix} \boxed{*} & \boxed{*} \\ 0 & \boxed{*} \end{pmatrix} \longmapsto x$$

$$\mu(x) = \begin{pmatrix} \boxed{*} & \boxed{*} \\ 0 & \boxed{*} \end{pmatrix} \in \mathfrak{gl}_{d,e}$$

$\circlearrowright$ $GL_d \times GL_e$ $\qquad$ $\circlearrowright$ $GL_{d,e}$ $\qquad$ $\circlearrowright$ GL_{d+e}

<u>exercise</u>: $\oplus$ is a finite morphism.

$X_{\bar{Q},d,e}$ and $X_{\bar{Q},d} \times X_{\bar{Q},e} \times \mathfrak{n}_{d,e}$ are smooth

and $q_{d,e}$ is of codimension

$$\dim X_{\bar{Q},d} + X_{\bar{Q},e} + \dim \mathfrak{n}_{d,e} - \dim X_{\bar{Q},d,e} + \dim P_{d,e} - \dim GL_d - \dim GL_e$$

$$= -\cancel{\langle d,d\rangle_{\bar{Q}}} - \cancel{\langle e,e\rangle_{\bar{Q}}} + d\cdot e + \cancel{\langle d,d\rangle_{\bar{Q}}} + \cancel{\langle e,e\rangle_{\bar{Q}}} + \langle d,e\rangle_{\bar{Q}}$$

$$= \langle d,e\rangle_Q + \langle e,d\rangle_Q$$
$$= (d,e)_Q \qquad \text{symmetrized Euler form of } Q.$$

and $p'_{d,e}$ is proper.

<u>Therefore</u>: $\quad q_{d,e}^{\ast}\, \mathbb{D}\mathbb{Q}_{X_{\bar{Q},d}\times X_{\bar{Q},e}\times \mathfrak{R}_{d,e}/GL_d\times GL_e} \cong \mathbb{D}\mathbb{Q}_{X_{\bar{Q},d,e}/P_{d,e}}[+2(d,e)_Q]$

and so

$$\mathbb{D}\mathbb{Q}_{X_{\bar{Q},d}\times X_{\bar{Q},e}\times \mathfrak{R}_{d,e}/GL_d\times GL_e} \longrightarrow (q_{d,e})_{\ast}\, \mathbb{D}\mathbb{Q}_{X_{\bar{Q},d,e}/P_{d,e}}[+2(d,e)_Q]$$

We apply $(i_d\times i_e)^!$ to obtain, by base-change, a morphism

$$\boxed{(\ast) \quad \mathbb{D}\mathbb{Q}_{\mu_d^{-1}(0)/GL_d} \boxtimes \mathbb{D}\mathbb{Q}_{\mu_e^{-1}(0)/GL_e} \longrightarrow (q'_{d,e})_{\ast}\, \mathbb{D}\mathbb{Q}_{\mu_{d,e}^{-1}(0)/GL_{d,e}}[2(d,e)_Q]}$$

Since $p'_{d,e}$ is proper, the pullback morphism

$$\mathbb{Q}_{\mu_{d+e}^{-1}(0)/GL_{d+e}} \longrightarrow (p'_{d,e})_{\ast}\mathbb{Q}_{\mu_{d,e}^{-1}(0)/GL_{d,e}} \qquad \text{dualizes}$$

to $\boxed{(\ast\ast) \quad (p'_{d,e})_{\ast}\, \mathbb{D}\mathbb{Q}_{\mu_{d,e}^{-1}(0)/GL_{d,e}} \longrightarrow \mathbb{D}\mathbb{Q}_{\mu_{d+e}^{-1}(0)/GL_{d+e}}}$

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virtual shifts It is customary to give the constant sheaf

the stack $\mu_d^{-1}(0)/GL_d$ the virtual shift:

$$\mathbb{Q}^{vir}_{\mu_d^{-1}(0)/GL_d} := \mathbb{Q}_{\mu_d^{-1}(0)/GL_d}[-(d,d)_Q]$$

Then, composing

$$(\oplus_{d,e})_* (\pi_d \times \pi_e)_* \left[(d,d)_Q + (e,e)_Q \right] \quad (*)$$

and $(\pi_{d+e})_*$ $(**)$ gives a morphism

$$\boxed{(\pi_d)_* \mathbb{D}\mathbb{Q}^{vir}_{\mu_d^{-1}(0)/GL_d} \boxtimes (\pi_e)_* \mathbb{D}\mathbb{Q}^{vir}_{\mu_e^{-1}(0)/GL_e} \xrightarrow{m_{d,e}} (\pi_{d+e})_* \mathbb{D}\mathbb{Q}^{vir}_{\mu_{d+e}^{-1}(0)/GL_{d+e}}}$$

Monoidal structure

$$\mathcal{M}_{\Pi_Q} := \bigsqcup_{d \in \mathbb{N}^{Q_0}} \mathcal{M}_{\Pi_Q, d}$$

∞ union of finite type affine schemes $/\mathbb{C}$.

monoid structure:

$$\oplus : \mathcal{M}_{\Pi_Q} \times \mathcal{M}_{\Pi_Q} \longrightarrow \mathcal{M}_{\Pi_Q}$$

finite morphism.

It induces a monoidal structure on $\mathcal{D}_c(\mathcal{M}_{\Pi_Q})$:

$$\mathcal{F}, \mathcal{G} \in \mathcal{D}_c(\mathcal{M}_{\Pi_Q}) \rightsquigarrow \mathcal{F} \boxdot \mathcal{G} := \oplus_* (\mathcal{F} \boxtimes \mathcal{G}).$$

Monoidal unit: $\mathbb{Q}_{\mathcal{M}_{\Pi_Q,0}}$ where $0 \in \mathbb{N}^{Q_0}$ so that $\mathcal{M}_{\Pi_Q,0} \cong pt$.

Algebra object : $\mathcal{A} \in \mathcal{D}_c(\mathcal{M}_{\Pi_\mathbb{Q}})$,

$$m : \mathcal{A} \boxdot \mathcal{A} \longrightarrow \mathcal{A}$$

$$\eta : \mathbb{Q}_{\mathcal{M}_{\Pi_\mathbb{Q},0}} \longrightarrow \mathcal{A}$$

satisfying the natural associativity and unitality axioms

$$\begin{tikzcd}
\mathcal{A} \boxdot \mathcal{A} \boxdot \mathcal{A} \arrow[r, "m \boxdot \mathrm{id}_\mathcal{A}"] \arrow[d, "\mathrm{id}_\mathcal{A} \boxdot m"'] & \mathcal{A} \boxdot \mathcal{A} \arrow[d, "m"] \\
\mathcal{A} \boxdot \mathcal{A} \arrow[r, "m"'] & \mathcal{A}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{A} \simeq \mathcal{A} \boxdot \mathbb{Q}_{\mathcal{M}_{\Pi_\mathbb{Q},0}} \arrow[r, "\mathrm{id}_\mathcal{A} \boxdot \eta"] \arrow[dr, "\mathrm{id}_\mathcal{A}"'] & \mathcal{A} \boxdot \mathcal{A} \arrow[d, "m"] \\
& \mathcal{A}
\end{tikzcd}$$

and similarly for $\eta \boxdot \mathrm{id}_\mathcal{A}$.

Sheafified cohomological Hall algebra of Π_Q

$$\mathcal{A}_{\Pi_Q} := \bigoplus_{d \in \mathbb{N}^{Q_0}} (\pi_d)_* \mathbb{D}\mathbb{Q}^{\mathrm{vir}}_{\mu_d^{-1}(0)/GL_d}$$

$$\in \mathcal{D}_c(\mathcal{M}_{\Pi_Q}).$$

Absolute cohomological Hall algebra

$p : \mathcal{M}_{\Pi_Q} \longrightarrow pt$

$$\begin{array}{l} p_* \mathcal{A}_{\Pi_Q} = \bigoplus_{d \in \mathbb{N}^{Q_0}} \underbrace{H^*\left(\mu_d^{-1}(0)/GL_d, \left(\mathbb{D}\mathbb{Q}_{\mu_d^{-1}(0)/GL_d}\right)[-\mathrm{vir}]\right)}_{H^{BM}_{\mathrm{vir}-*}(\mu_d^{-1}(0)/GL_d)} \\ \quad \Vert \\ A_{\Pi_Q} \end{array}$$

$$\mathrm{vir} = -(d,d)_Q$$

$$A_{\Pi_Q,d,i} \otimes A_{\Pi_Q,e,j} \longrightarrow A_{\Pi_Q,d+e,i+j}$$

(co)homological degree

bigraded algebra.