Colloquium Dublin

30 mins for non-specialists.

BPS algebra of a 2CY category

joint w/ Ben Davison & Sebastian S.M.

BPS algebras of 2CY categories (0-dimensional sheaves on a surface)

related to cohomological Hall algebras

~ homological condition

parametrising spaces

Goal: Studying BM homology of the stack of objects of categories arising in

* rep theory: preprojective algebras of quivers (multiplicative)
* geometry: sheaves on symplectic surfaces
  0-dim sheaves on any surface.
* topology: fundamental group algebra of a (hyperbolic) Riemann surface.

Result: Uniform behaviour.

Motivations: ① interesting in itself: understanding Betti numbers of various moduli spaces.

② interesting related conjectures (now theorems)
- nonabelian Hodge isos for stacks
- positivity of cusp. pols of quivers
- decomposition of the cohomology of NQV.

Plan

* Coh. integrality
* integrality for 2CY categories
* GKM algebras in some tensor categories
* CoHAs and BPS algebras
* Ingredients of the proof: local neighbourhood thm
  top SSN CoHA.

Cohomological integrality for 2CY categories

Dublin – 30 min for non-specialists.

joint with Ben Davison and Sebastián Schlegel Mejia.

arXiv: 2212.07668

arXiv: 2303.12592

topology and en. invariants (Betti #)

we can break these into smaller well-behaved pieces.

symplectic aspect.

S symp surface

sheaves on S → parametrizing spaces

Goal: Studying the BM homology of the stack of objects of categories arising in
≃ topology

a) rep theory : (multiplicative) preprojective algebras of quivers

b) geometry : sheaves on symplectic surfaces, 0-dim sheaves on any surface.

c) topology : fundamental group algebra of a Riemann surface.

Result: Uniform behaviour, described in terms of generalised Kac-Moody algebras; many interactions between a), b), c).

Motivations: interesting related conjectures

* nonabelian Hodge isomorphisms for stacks D, DHS
* positivity of cuspidal polynomials of quivers DHS
* decomposition of the cohomology of NQV. DHS
* CoHA upgrade of NAHT for stacks : H, proving a refinement of a conjecture expressed for categorified HA by Porta-Sala

All these are now theorems

* More to come in the near future.

9 (Cohomological) integrality add $q^{1/2}$.

$M \subset \mathbb{N}^S$ S a finite set
monoid. $0 \in M$, M stable under addition

$$\mathbb{Q}(q^{1/2})[[M]] = \left\{ a = \sum_{m \in M} a_m z^m : a_m \in \mathbb{Q}(q^{1/2}) \right\}$$

$$\cup$$

$$\mathbb{Q}(q^{1/2})[[M]]_+ = \{ a : a_0 = 0 \}$$

Plethystic operations:

$$\mathrm{Exp}_{q,z} : \mathbb{Q}(q^{1/2})[[M]]_+ \longrightarrow 1 + \mathbb{Q}(q^{1/2})[[M]]_+$$

- multiplicative: $\mathrm{Exp}_{q,z}(a+b) = \mathrm{Exp}_{q,z}(a)\,\mathrm{Exp}_{q,z}(b)$
- $\mathrm{Exp}_{q,z}(q^n z^m) = \dfrac{1}{1-q^n z^m}$

* Can be defined very explicitly in terms of the λ-ring structure of $\mathbb{Q}(q^{1/2})[[M]]$

* It has an inverse $\mathrm{Log}_{q,z} : 1 + \mathbb{Q}(q^{1/2})[[M]]_+ \to \mathbb{Q}(q^{1/2})[[M]]_+$ also very explicit.

<u>integrality</u>: $a \in 1 + \mathbb{Q}(q^{1/2})[[M]]_+$ satisfies integrality if

$$a \in \mathrm{Exp}_{q,z}\left(\frac{\mathbb{Z}[q^{\pm 1/2}][[M]]_+}{1-q} \right)$$

i.e. $\mathrm{Log}_{q,z}(a) \in \dfrac{\mathbb{Z}[q^{\pm 1/2}][[M]]_+}{1-q}$.

sometimes $q^{1/2} - q^{-1/2}$

<u>categorified integrality</u>: If $V = \bigoplus_{m \in M} V_m$ is a $M \times \mathbb{Z}$ - graded vector space, $\dim V_0 = 0$, $\mathrm{ch}\, V := \sum_{m \in M} \underset{\substack{\cap \\ \mathbb{N}[q^{\pm 1/2}]}}{(\dim V_m)} z^m \in \mathbb{N}[q^{\pm 1/2}][[M]]_+$,

$$\mathrm{ch}\Big(\mathrm{Sym}\big(\underset{\substack{z,q \text{ degree} = \\ 0,2}}{V\otimes \mathbb{C}[u]}\big)\Big) = \mathrm{ch}\Big(\bigoplus_{n\geq 0}\big((V\otimes\mathbb{C}[u])^{\otimes n}\big)^{\mathfrak{S}_n}\Big)$$

$$= \mathrm{Exp}_z\left(\frac{\mathrm{ch}(V)}{1-q}\right).$$

So let's say that a M-graded vector space W satisfies integrality if there exists a vector space V as above s.t. $W \cong \mathrm{Sym}(V\otimes\mathbb{C}[u])$. V is called the BPS vector space (Bogomol'nyi Prasad Sommerfield.) W is the DT vector space.

More generally, if $\mathcal{A}$ is a $M\times\mathbb{Z}$-graded tensor category, one can mimic the same definition.

* Occurrences of integrality: ① Counting reps of quivers / finite fields

② Symmetric quivers: conjectures by Kontsevich-Soibelman, theorem of Efimov ("1CY" situation)

$$\bigoplus_{d\in\mathbb{N}^{Q_0}} H^*(pt/GL_d) \cong \mathrm{Sym}\Big(\bigoplus_{d\in\mathbb{N}^{Q_0}} \mathrm{BPS}_d \otimes \mathbb{C}[u]\Big)$$

②' Semistable vector bundles on surfaces: Mozgovoy - Reineke.

③ Symmetric quivers w/ potential + categorifications: Davison - Meinhardt (3CY situation)

Categorified versions: Meinhardt - Reineke (symmetric quivers)

(use the λ-ring structure on some Grothendieck group of MHM). (categorified 1CY)

(4) Quivers and twisted Higgs bundles (counting)

* Mozgovoy – Schiffmann, Mozgovoy O'Gorman

(5) Surfaces (Fano condition with polarization and) Manschot – Mozgovoy

Mostly from 1 situations (smooth moduli stacks) as 3CY situations

2CY situations w/ counts of Higgs bundles for example.

Today: More general 2CY situations and cohomology

② $\underline{\text{Today}}$: (categorified) $\underline{\text{2CY versions}}$ of cohomological integrality.

$\mathcal{A}$ an Abelian category satisfying the necessary assumptions

[categorical nature $\mathcal{A} \subset \mathcal{E}$ dg cat.
geometric nature $\mathfrak{M}_{\mathcal{A}} \xrightarrow{JH} \mathcal{M}_{\mathcal{A}}$ good moduli space.
and categorico-geometric nature]

$\underline{\text{e.g.}}$ ① $\mathcal{A} = \operatorname{Rep} \Pi_Q$

② $\mathcal{A} = \operatorname{Coh}^{H\text{-}ss}_{p(t)}(S)$ $\quad$ S symplectic surface

③ $\mathcal{A} = \operatorname{Coh}_0(S)$ $\quad$ length 0 sheaves on any surface

④ $\mathcal{A} = \operatorname{Rep} \Lambda_Q$ $\quad$ multiplicative preprojective algebra

⑤ $\mathcal{A} = \operatorname{Rep} \pi_1(S, x)$ $\quad$ S genus ≥ 1 Riemann surface.

$\mathfrak{M}_{\mathcal{A}}$ the stack of objects of $\mathcal{A}$.

$\downarrow JH$ the good moduli space.

$\mathcal{M}_{\mathcal{A}}$

[Other stacks/good moduli spaces have a geometry of the same flavour].

$\underline{\text{e.g. for } \boxed{\Pi_Q}}$: $Q = (Q_0, Q_1)$ quiver
(Q_0 vertices, Q_1 arrows)

$i \xrightarrow{\alpha} j$

$\overline{Q}$ its double $= (Q_0, \overline{Q}_1)$

$i \underset{\alpha^*}{\overset{\alpha}{\rightleftarrows}} j$

$$\Pi_Q = \mathbb{C}\overline{Q} \Big/ \sum_{\alpha \in Q_1} [\alpha, \alpha^*]$$

Stack of representations:

$$d \in \mathbb{N}^{Q_0} \qquad X_{\overline{Q},d} = T^*\Big(\bigoplus_{\alpha \in Q_1} \operatorname{Hom}(\mathbb{C}^{d_i}, \mathbb{C}^{d_j})\Big) \circlearrowleft GL_d = \prod_{i \in Q_0} GL_{d_i}$$

in a Hamiltonian fashion

$$\mu_d : X_{\bar{Q},d} \longrightarrow \mathfrak{gl}_d \cong \mathfrak{gl}_d^*$$

$$(x_\alpha, x_{\alpha^*})_{\alpha \in Q_1} \longmapsto \sum_{\alpha \in Q_1} [x_\alpha, x_{\alpha^*}].$$

$\mathfrak{M}_{\Pi_Q,d} = \mu_d^{-1}(0)/GL_d$ quotient stack

singular, reducible, not equidimensional: this is a terrible variety!

$$\begin{array}{c} \mathfrak{M}_{\Pi_Q,d} \\ \text{JH gms} \big\downarrow \\ \mathcal{M}_{\Pi_Q,d} \end{array}$$

$\mathcal{M}_{\Pi_Q,d} = \mu_d^{-1}(0)/\!/GL_d$ GIT quotient

$= \mathrm{Spec}\left(\mathbb{C}[\mu_d^{-1}(0)]^{GL_d}\right).$

e.g. $\pi_1(S, x)$: similar but use of multiplicative moment map.

Borel-Moore homology:

$$\mathcal{A} := \mathrm{JH}_* \, \mathbb{D}\mathbb{Q}^{\mathrm{vir}}_{\mathfrak{M}_{\mathcal{A}}} \in \mathcal{D}^+_c(\mathcal{M}_{\mathcal{A}}) \Big/ \mathcal{D}^+(\mathrm{MHM}(\mathcal{M}_{\mathcal{A}}))$$

coh. shift

$H^{\mathrm{BM}}_{*+\mathrm{vir}}(\mathfrak{M}_{\mathcal{A}})$ (shifted) BM-homology of $\mathfrak{M}_{\mathcal{A}}$.

$$\| \\ H^{-*}(\mathcal{A})$$

$M \times \mathbb{Z}$-graded, $M = \pi_0(\mathfrak{M}_{\mathcal{A}})$.

Thm [DHS]

1. $H^{\mathrm{BM}}_{*+\mathrm{vir}}(\mathfrak{M}_{\mathcal{A}}) \cong \mathrm{Sym}(\mathrm{BPS} \otimes H^*(B\mathbb{C}^*))$ satisfies cohomological integrality
2. Coh. integrality can be lifted to $\mathcal{A} \in \mathcal{D}^+_c(\mathcal{M}_{\mathcal{A}})$
3. $H^{\mathrm{BM}}_{*+\mathrm{vir}}(\mathfrak{M}_{\mathcal{A}})$ has an algebra structure
4. The BPS space has a Lie algebra structure
5. The cohomological integrality isomorphism
$$\mathrm{Sym}\left(\mathrm{BPS} \otimes H^*(B\mathbb{C}^*)\right) \to H^{\mathrm{BM}}_{*+\mathrm{vir}}(\mathfrak{M}_{\mathcal{A}})$$
is of PBW-type.

3, 4, 5 can be lifted to $\mathcal{D}^+_c(\mathcal{M}_{\mathcal{A}}) / \mathcal{D}^+(\mathrm{MHM}(\mathcal{M}_{\mathcal{A}}))$

6. BPS or its lift $\mathcal{BPS}$ are generalised Kac-Moody algebras, explicitly described.

In particular, using this description, BPS is generated by $IH(\mathcal{M}_{\mathcal{A},a})$, $a \in R^+_{\mathcal{A}} \subset \pi_0(\mathfrak{M}_{\mathcal{A}})$ certain set of connected components

The results are proven for the lifts first, and deduced for the actual algebras by taking derived global sections.

Now: introduce CoHA formalism to study $\mathcal{A}$; $H^*\mathcal{A}$.

③ (a) CoHA structure on $\mathcal{A}$
(b) GKM algebras in some tensor categories

(a) $M \subset \mathbb{N}^S$ a monoid

$\mathcal{M} = \bigsqcup_{m \in M} \mathcal{M}_m$ a complex scheme, with monoid structure

$$\begin{array}{ccc} \mathcal{M} \times \mathcal{M} & \xrightarrow{\oplus} & \mathcal{M} \\ \downarrow & \circlearrowleft & \downarrow \\ M \times M & \longrightarrow & M \end{array}$$

Symmetric : $\oplus \circ sw = \oplus$.

$\mathcal{D}_c^+(\mathcal{M})$ becomes a symmetric monoidal category :

$$\mathcal{F} \boxdot \mathcal{G} := \oplus_*(\mathcal{F} \boxtimes \mathcal{G}).$$

Assume $\mathcal{M}_0 = pt$. (simplicity)

If $\oplus$ is finite, $\oplus_*$ is perverse t-exact and so we get a symmetric tensor category $(\mathrm{Perv}(\mathcal{M}), \boxdot)$.

monoidal unit: $\mathbb{1} = \mathbb{Q}_{\mathcal{M}_0}$

- algebra objects : $(\mathcal{A}, m)$,

  $m : \mathcal{A} \boxdot \mathcal{A} \longrightarrow \mathcal{A}$,

  $\mathbb{1} \longrightarrow \mathcal{A}$ unit.

  satisfying usual axioms (associativity).

- Lie algebra objects : $(\mathcal{L}, [-,-])$ $\quad [-,-] : \mathcal{L} \boxdot \mathcal{L} \longrightarrow \mathcal{L}$.

Need symmetric tensor category.

Generalised Kac-Moody Lie algebras

$(-,-): M \times M \longrightarrow \mathbb{Z}$ bilinear form.

Roots: choose $R \subset M$ (potentially infinite) s.t.

$$\big((r,s)\big)_{r,s\in R} \quad \text{is a generalised Cartan matrix}$$

i.e. $\begin{cases} (r,r) \in 2\mathbb{Z}_{\leq 2} \\ (r,s) \in \mathbb{Z}_{\leq 0} & \forall r \neq s \end{cases}$

Let $\mathcal{F}_r \in \mathrm{Perv}(\mathcal{M})$, $r \in R$, $\mathcal{F} = \bigoplus_{r\in R} \mathcal{F}_r$.

$$\mathrm{Free}(\mathcal{F}) := \bigoplus_{r\geq 0} \mathcal{F}^{\boxdot r} \quad \text{Free algebra}$$

$\cup$

$\mathrm{FreeLie}(\mathcal{F})$ subobject containing $\mathcal{F}$ and stable under $[-,-]$.

Serre ideal: $\mathcal{I}_{\mathcal{F}}$ (Lie ideal) generated by

$$\begin{cases} [\mathcal{F}_r, \mathcal{F}_s] & \text{if } r,s = 0 \\ \mathrm{ad}(\mathcal{F}_r)^{1-(r,s)}(\mathcal{F}_s) & \text{if } r \neq s \end{cases}$$

$\mathfrak{n}^+_{\mathcal{F}} := \mathrm{FreeLie}(\mathcal{F}) / \mathcal{I}_{\mathcal{F}}$ is a half GKM, generated by $\mathcal{F}$.

$\in \mathrm{Perv}(\mathcal{M})$.

$H^*(\mathfrak{n}^+_{\mathcal{F}})$ derived global sections recovers (positive parts of) GKM in a more classical sense.

<u>enveloping algebras</u> $(\mathcal{L}, [-,-]) \in (\mathrm{Perv}(\mathcal{M}), \odot)$ Lie alg.

* $U(\mathcal{L})$ enveloping algebra
* PBW-theorem $\mathrm{Sym}(\mathcal{L}) \cong U(\mathcal{L})$.

for GKM Lie algebras: $U(\mathfrak{n}^+_{\mathcal{F}}) = \mathrm{Free}(\mathcal{F}) / \langle\langle \mathcal{J}_{\mathcal{F}} \rangle\rangle$.

Everything works as for vector spaces.

(a) Cohomological Hall algebras

Recall: want an algebra structure on $JH_* \mathbb{D}\mathbb{Q}^{vir}_{\mathfrak{M}_{\mathcal{A}}}$.

has a virtual dimension.

$$\begin{tikzcd}
\mathfrak{M}_{\mathcal{A}} \times \mathfrak{M}_{\mathcal{A}} \arrow[d, "JH \times JH"'] & \mathrm{Exact}_{\mathcal{A}} \arrow[l, "q"'] \arrow[r, "p"] \arrow[d, phantom, "\circlearrowleft"] & \mathfrak{M}_{\mathcal{A}} \arrow[d, "JH"] \\
\mathcal{M}_{\mathcal{A}} \times \mathcal{M}_{\mathcal{A}} \arrow[rr, "\oplus"'] & & \mathcal{M}_{\mathcal{A}}
\end{tikzcd}$$

(1) * p is proper

(2) * q is presented in a very favourable way, allowing one to define the pull-back by q in BM homology.

(1) $p_! = p_*$ $\qquad p^* \mathbb{Q}_{\mathfrak{M}_{\mathcal{A}}} = \mathbb{Q}_{\mathrm{Exact}_{\mathcal{A}}}$

$$\mathbb{Q}_{\mathfrak{M}_{\mathcal{A}}} \longrightarrow p_* \mathbb{Q}_{\mathrm{Exact}_{\mathcal{A}}}$$

$$p_! \mathbb{D}\mathbb{Q}_{\mathrm{Exact}_{\mathcal{A}}} = p_* \mathbb{Q}_{\mathrm{Exact}_{\mathcal{A}}} \longrightarrow \mathbb{D}\mathbb{Q}_{\mathfrak{M}_{\mathcal{A}}}$$

p.f at the sheaf level.

(2)

$$\begin{tikzcd}
 & F \arrow[d, "v.b"] \\
\mathrm{Exact}_{\mathcal{A}} = s^{-1}(0) \arrow[r, hook] & E \arrow[d, "v.b."] \arrow[u, bend right, "s"'] \\
 & \mathfrak{M}_{\mathcal{A}} \times \mathfrak{M}_{\mathcal{A}}
\end{tikzcd}$$

s section.

→ Can define the pullback at the sheaf level

$$\mathbb{D}\mathbb{Q}_{\mathfrak{M}_{\mathcal{A}} \times \mathfrak{M}_{\mathcal{A}}} \longrightarrow q_* \mathbb{D}\mathbb{Q}_{\mathrm{Exact}_{\mathcal{A}}}[2 \mathrm{vir}\, q]$$

$\rightarrow$ $m : \mathcal{DQ}^{vir}_{\mathfrak{M}_\mathcal{A} \times \mathfrak{M}_\mathcal{A}} \longrightarrow \mathcal{DQ}^{vir}_{\mathfrak{M}_\mathcal{A}}$ multiplication.

<u>Thm</u> (DHS) We get an associative algebra structure on $\mathcal{A}$.

BPS algebras: The BPS <u>Lie</u> algebra is not (yet) geometrically accessible yet. [for general 2CY categories]

We define the BPS algebra as

$$BPS_\mathcal{A} = {}^p\mathcal{H}^0(\mathcal{A}).$$

<u>Davison</u>: $\mathcal{A}$ is concentrated in ≥ 0 perverse degrees

$\Rightarrow$ $BPS_\mathcal{A}$ has an induced algebra structure

<u>Roots</u>: $\pi_0(\mathfrak{M}_\mathcal{A}) =: M$

$(-,-) : M \times M \longrightarrow \mathbb{Z}$ Euler form

$$R^+_\mathcal{A} = \Sigma_\mathcal{A} \cup \{ lr : \underset{l \geq 2}{r \in \Sigma_\mathcal{A}}, (r,r) = 0 \}$$

where $\Sigma_\mathcal{A} = \{ m \in M \mid \forall$ nontrivial decomposition
$m = \sum_i m_i,$
$2 - (m,m) > \sum_i (2 - (m_i, m_i))$
(Crawley-Boevey condition)$\}$.

<u>Generators</u> if $r \in \Sigma_\mathcal{A}$, $\mathcal{F}_r = \mathcal{IC}(\mathcal{M}_{\mathcal{A},r})$

if $r \in \Sigma_\mathcal{A}$, $l \geq 2$, $\mathcal{M}_{\mathcal{A},r} \xrightarrow{\Delta} \mathcal{M}_{\mathcal{A},lr}$
$x \longmapsto x^{\oplus l}$

$$\mathcal{F}_{\ell_r} = \Delta_* \mathcal{IC}(\mathcal{M}_{A,r}).$$

<u>Thm</u> (A) $BPS_{A} \simeq \mathcal{H}(\mathcal{M}^+_{\mathcal{F}})$.

<u>Definition</u>: $BPS_{Lie,A} := \mathcal{M}^+_{\mathcal{F}} \in \mathrm{Perv}(\mathcal{M}_A)$.

<u>Thm</u> (B) $BPS_{Lie,A}$ gives coh. integrality for A:

$$\mathrm{Sym}_{\boxdot}\left(BPS_{Lie,A} \otimes H^*_{\mathbb{C}^*}\right) \underset{\text{cstble complexes}}{\cong} A.$$

<u>examples</u>: finite length coh sheaves on surfaces.

S a smooth quasi-projective surface.

e.g. $S = \mathbb{A}^2$

$\mathrm{Coh}_0(S)$ is an Abelian category

* simple objects: $\mathbb{C}_x$ skyscraper sheaf supported at $x \in S$.
* if $x_i \in S$ distinct,

  $\langle \mathbb{C}_{x_i} \rangle$ has a right 2CY structure.

  vanishing Euler form.

* $\mathcal{M}_{\mathrm{Coh}_0(S)} = \coprod_{\ell} \mathcal{M}_{\mathrm{Coh}_{0,\ell}(S)}$

  $\ell = 1$: $\mathcal{M}_{\mathrm{Coh}_{0,\ell}(S)} \cong S/\mathbb{C}^\times$

$\ell > 1$: $\mathfrak{M}_{Coh_{0,\ell}(S)} \supset \dfrac{Sym^\ell(S) \setminus \Delta^{big}}{(\mathbb{C}^*)^\ell}$ open.

$$\begin{tikzcd} \mathfrak{M}_{Coh_{0,\ell}(S)} & \mathbb{C}_x^{\oplus \ell} \\ S/\mathbb{C}^* \arrow[u] & x \arrow[u, "\Delta^{sm}"'] \end{tikzcd}$$

$S/\mathbb{C}^* \ni x$, $\mathcal{M}_{Coh_{0,1}(S)} \cong S$

<u>Theorem B</u> $\Rightarrow$ $H^{BM}_*(\mathfrak{M}_{Coh_0(S)}) \cong Sym\left(\bigoplus_{\ell \geq 1} \mathcal{IC}(S) \otimes H^*_{\mathbb{C}^*}\right)$

<u>no virtual shift as the Euler form vanishes</u>

In these case, we identify

$BPS_{\mathcal{A},Lie} = \bigoplus_{\ell \geq 1} \mathcal{IC}(S)$ w/ the 0-Lie bracket

$BPS_{\mathcal{A}} = Sym_{\boxminus}(BPS_{\mathcal{A},Lie})$ as algebras $\to$ $\oplus$ of local systems coming from symmetric groups.

<u>Proof</u>: This is an intricate induction relying on 2 essential results:

1. <u>Davison's local neighbourhood theorem for 2CY categories</u>
2. <u>Identification of the top-strictly seminilpotent CoHAs of quivers.</u> (H)

$\boxed{1}$ $\mathcal{A}$ 2CY $\mathfrak{M}_{\mathcal{A}} \xrightarrow{JH} \mathcal{M}_{\mathcal{A}} \ni x$

x corresponds to $\mathcal{F} = \bigoplus_{i=1}^{r} \mathcal{F}_i^{m_i}$.

$\underline{\mathcal{F}} = \{\mathcal{F}_i, 1 \leq i \leq r\}$ collection of simple of $\mathcal{A}$

$\overline{Q}_{\underline{\mathcal{F}}} = (\underline{\mathcal{F}}, \text{arrows})$

$\#\{\mathcal{F}_i \to \mathcal{F}_j\} = \mathrm{ext}^1(\mathcal{F}_i, \mathcal{F}_j)$.

$\overline{Q}_{\underline{\mathcal{F}}}$ is the double of some (non-unique) quiver $Q_{\underline{\mathcal{F}}}$.

$$\begin{tikzcd}
(\mathfrak{M}_{\mathcal{A}}, x) \arrow[d, "JH"'] & (\mathcal{X}, y) \arrow[l] \arrow[r] \arrow[d] \arrow[l, phantom, "\ulcorner"] \arrow[r, phantom, "\urcorner"] & (\mathfrak{M}_{\Pi_Q}, 0_m) \arrow[d, "JH"] \\
(\mathcal{M}_{\mathcal{A}}, x) & (\mathcal{U}, y) \arrow[l] \arrow[r] & (\mathcal{M}_{\Pi_Q}, 0_m)
\end{tikzcd}$$

with étale horizontal maps.

$\boxed{2}$ $Q = (Q_0, Q_1)$

Lagrangian substack.

$$\begin{tikzcd}
\mathfrak{M}_{\Pi_Q}^{SSN} \arrow[r, "i"] \arrow[d, "JH^{SSN}"'] \arrow[dr, phantom, "\ulcorner"] & \mathfrak{M}_{\Pi_Q} \arrow[d, "JH"] \\
\mathcal{M}_{\Pi_Q}^{SSN} \arrow[r, hook, "i", "\text{submonoid}"'] & \mathcal{M}_{\Pi_Q}
\end{tikzcd}$$

$\mathcal{M}_{\Pi_Q}^{SSN} = \left\{ (x_\alpha, x_{\alpha^*}) \in \mathcal{M}_{\Pi_Q} \;\middle|\; \begin{array}{l} x_\alpha = 0 \text{ and } x_{\alpha^*} = 0 \text{ if} \\ \alpha^* \text{ is not a loop} \end{array} \right\}$

$$H^*(i^! \mathcal{A}_{\Pi_Q}) =: H^*(\mathcal{A}^{SSN}_{\Pi_Q}) \supset H^0(\mathcal{A}^{SSN}_{\Pi_Q})$$
subalgebra.

$H^0(\mathcal{A}^{SSN}_{\Pi_Q})$ has a $\mathbb{C}$-linear basis given by irreducible components of $\mathfrak{M}^{SSN}_{\Pi_Q}$. [Bozec]

Thm [H] $H^0(\mathcal{A}^{SSN}_{\Pi_Q}) \cong U(\mathfrak{n}^+_Q)$ where $\mathfrak{n}^+_Q$ is Bozec's Lie algebra of Q

[when Q has no loops, this is the KM algebra of Q].

if loops, need to take them into account.