Elliptic Quantum Groups

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(joint work in progress w/ Y. Yang and G

§1: Motivation

Let E be an elliptic curve $/\mathbb{C}$

G reductive group

$$\mathrm{Bun}_G = \mathrm{Bun}_G(E) = \left\{ \begin{array}{c} \text{principal } G\text{-bundles} \\ \text{on } E \end{array} \right\}$$

on Artin stack

Then [PTVV]:

$$\left\{ \begin{array}{l} \text{nondegenerate} \\ (\,,\,) \in \mathrm{Sym}^2(\mathfrak{g}^*)^G \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{1-shifted symplectic} \\ \text{forms } \omega \text{ on } \mathrm{Bun}_G \end{array} \right\}$$

Formality [CPTVV]: For each ω

$\exists$ monoidal deformation $(\mathrm{QCoh}_\hbar(\mathrm{Bun}_G), \otimes_\hbar)$ of $(\mathrm{QCoh}($

over $\mathbb{C}[[\hbar]]$. - as $\mathbb{C}[[\hbar]]$-linear category.

Ultimate goal: Construct $(\mathrm{QCoh}_\hbar(\mathrm{Bun}_G), \otimes_\hbar)$ explicitly for $\hbar \in E$.

How to quantise? Choose $\emptyset \neq P \subset |E|$ closed points set

$$K = \Gamma(E - P, \mathcal{O})$$

$$\mathbb{O} = \prod_{x \in P} \hat{\mathcal{O}}_{E,x}$$

$$\mathbb{A} = \prod_{x\in P}' \mathrm{Frac}(\hat{\mathcal{O}}_{E,x})$$

For G semisimple, then

$$\mathrm{Bun}_G = G(K) \backslash G(\mathbb{A}) / G(\mathbb{O}) = BG(K) \times_{BG(\mathbb{A})} BG($$

So $\mathrm{QCoh}(\mathrm{Bun}_G)$ "=" $\mathrm{Rep}\, G(K) \otimes_{\mathrm{Rep} G(\mathbb{A})} \mathrm{Rep}\, G(\mathbb{O})$

Intermediate goal: Construct quantisations of

$\rightsquigarrow$ Quantise Hopf algebras

$U(\mathfrak{g}_K)$, $U(\mathfrak{g}_{\mathbb{O}})$, $U(\mathfrak{g}_{\mathbb{A}})$.

Outline: 2: History

3: Main results/constructions
4: Positive part
5: Double construction
6: Dynamical representations

§2: Incomplete history of elliptic quantum groups

Traditionally,

elliptic quantum groups = Hopf algebras (or simila... associated w/ elliptic s... to Yang-Baxter equation ...

1980's: [Belavin-Drinfeld]: classified sol^{n}s to the cl... valued in f.d. Lie alg

- Rational $\rightsquigarrow$ e.g. Yangians
- Trigonometric $\rightsquigarrow$ eg. quantum affine algeb...
- Elliptic $\rightsquigarrow$ elliptic quantum groups

Exist only in type A

[Sklyanin, Cherednik, Feigin-Odesskii]:
Algebras assoc. w/ type A elliptic solution...

[Felder]: Use dynamical YBE instead
$\rightsquigarrow$ elliptic sol^{n}s in all types

1990's - 2010's: Many approaches to elliptic quant... groups based on Felder's sol^{n}s.
(many authors)

Relevant ones for this talk:

[Gautam-Toledano Laredo '17]: Constructed + studied category of reps for any $\mathfrak{g}$.

- Vector space V + diagonalisable $\mathfrak{h}$-action (Cartan)
- Currents $\Phi_i(u,\lambda)$, $\mathcal{X}_i^{\pm}(u,\lambda) : V \longrightarrow V$

meromorphic in u, λ, satisfying Yangian-li...

Spectral dynamical

Key feature: dynamical parameters shift.

E.g.
$$\Phi_i(u, \lambda+\tfrac{\hbar}{2}\alpha_j)\,\Phi_j(v, \lambda-\tfrac{\hbar}{2}\alpha_i) =$$
$$\Phi_j(v, \lambda+\tfrac{\hbar}{2}\alpha_i)\,\Phi_i(u, \lambda-\tfrac{\hbar}{2}\alpha_j)$$

- [Yang–Zhao '17]:
  Elliptic CoHA preprojective algebra
  $\Rightarrow$ positive part of an elliptic quantum group
  $$\mathcal{E}ll_\hbar(\mathfrak{n}_+) \in (\mathrm{Coh}(H^+), \underset{\hbar}{\otimes})$$
  quantum monoid structure
  I-coloured Hilbert scheme of E

§ 3: Main results/constructions

Fix $\mathfrak{g}$ ADE Lie algebra, $I = \{$vertices of Dynkin diag$\}$

$\hbar/2 \in E$ generic, $\emptyset \neq P \subset |E|$ invariant under tran by $\hbar/2$ $\Rightarrow$ P is infinite

1. There exist algebra objects
$$\mathcal{E}ll_\hbar(\mathfrak{g}_A) \in (\mathrm{Shv}(H^+ \times H^-), \underset{\hbar}{\otimes})$$
"topological sheaves"

and subquotients
$$\mathcal{E}ll_\hbar(\mathfrak{g}_0),\ \mathcal{E}ll_\hbar(\mathfrak{g}_K).$$

2. There is an action
$$(\mathrm{Shv}(H^+ \times H^-), \underset{\hbar}{\otimes}) \circlearrowright \mathrm{Shv}(\mathrm{Bun}_T^\circ)$$
space of dy param

and notions of integrable reps of $\mathcal{E}ll_\hbar(\mathfrak{g}_\square)$ in

3. For $V \in \mathrm{Rep}^{int}\,\mathcal{E}ll_\hbar(\mathfrak{g}_0)$ and rational iso $V \cong$ alo

V carries grading + currents
$\Phi_i(u,\lambda)$, $\mathfrak{X}_i^\pm(u,\lambda)$ satisfy GTL relations.

§ 4: Positive part:

Construction: shuffle algebras

Let X be an (affine) curve, I finite set.

Let $H = \mathrm{Hilb}(X \times I) = \mathrm{Sym}(X \times I) = \coprod_{v \in \mathbb{Z}^I_{\geq 0}} X^{(v)}$, $X^{(v)} =$

= moduli space of effective coloured divisors

$$D = \sum_j w_j x_j, \quad w_j \in \mathbb{Z}^I_{\geq 0}.$$

Have $\mathcal{S} : H \times H \longrightarrow H$ sum of divisors (fini f

Let $\Delta = \{ (D, D') \mid D \cap D' \neq \emptyset \} \subset H \times H$

$= \mathrm{ram}(\mathcal{S})$

rat. functions w/ at simple poles

Suppose $\Omega(z|w) \in \Gamma(H \times H, \mathcal{O}_{H \times H}(\Delta))$ satisfies

$$\Omega(\mathcal{S}(z, z') | w) = \Omega(z|w)\Omega(z'|w)$$
$$\Omega(z | \mathcal{S}(w, w')) = \Omega(z|w)\Omega(z|w')$$

The <u>shuffle algebra</u> with kernel Ω is

$SH = SH_{\Omega} = \Gamma(H, \mathcal{O}_H)$ with product

$$f * g = \mathrm{Tr}_{\mathcal{S}}(\Omega(z|w) f(z) g(w))$$

trace = sum over preimages

= symmetriser

<u>Sheaf version</u>: Define a monoidal structure $*$

$$\mathcal{F} * \mathcal{G} = \mathcal{S}_*(\mathcal{F} \boxtimes \mathcal{G})$$

Then have sheafified shuffle algebra

$\mathcal{SH} = \mathcal{O}_H$ with product

$$\mathcal{O}_H * \mathcal{O}_H = \mathcal{S}_*(\mathcal{O}_{H \times H}) \longrightarrow \mathcal{S}_*(\mathcal{O}_{H \times H}(\Delta)) \xrightarrow{\mathrm{Tr}_{\mathcal{S}}} \mathcal{O}_H$$

$\Gamma : (\mathrm{Coh}(H), *) \longrightarrow (\mathrm{Vect}, \otimes)$ is monoidal. (

and $SH = \Gamma(\mathcal{SH})$.

Spherical subalgebras: $SH^{sph} = \Gamma(\mathcal{SH}^{sph})$ subalg
generated by $\bigoplus_i \mathcal{O}_{X^{(e_i)}}$.

Thm: (e.g. [Yang–Zhao]): Fix ADE Dynkin diag w
$X = \mathbb{A}^1 = \mathbb{C}$.

$$\Omega(z|w) = \prod_{j,k} \Omega(z_j|w_k) \text{ for}$$

$$\Omega(z_j|w_k) = \begin{cases} \dfrac{z_j - w_k + \hbar}{z_j - w_k} & \text{if } col(z_j) = col(w \\ \pm(z_j - w_k + \frac{\hbar}{2}) & \text{if } col(z_j) \text{ adjac} \\ 1 & \text{otherwise.} \end{cases}$$

then $SH^{sph} \cong Y_\hbar(n_+) \subset Y_\hbar(\mathfrak{g})$

$z^r \longmapsto \hbar\, x^+_{i,r}$ (Drinfeld new generator)

colour i

Elliptic case: Now let $X = E$ elliptic curve (not

Set
$$\Omega(z_j|w_k) = \begin{cases} \dfrac{\vartheta(z_j - w_k - \hbar)}{\vartheta(z_j - w_k)} & \text{----} \\ \pm\vartheta(z_j - w_k + \frac{\hbar}{2}) & \text{---} \\ 1 & \text{---} \end{cases}$$

where $\vartheta(z)$ is the unique section of $\mathcal{O}(0_E)$

$\therefore\ \Omega(z|w) = \prod_{j,k} \Omega(z_j|w_k)$ is a rational section of bundle $\mathcal{L}^{-1}$ on $\mathcal{H} \times \mathcal{H}$.

So $\Omega : \mathcal{L} \longrightarrow \mathcal{O}(\Delta)$.

Defn: For $\mathcal{F}, \mathcal{G} \in Coh(\mathcal{H})$, set

$$\mathcal{F} \otimes \mathcal{G} = \mathcal{B}_*(\mathcal{L} \otimes (\mathcal{F} \boxtimes \mathcal{G}))$$

quantum struct

$\hbar$

$\mathcal{SH} = \mathcal{O}_H$ with product

$$\mathcal{O}_H \otimes_\hbar \mathcal{O}_H = \mathcal{S}_*(\mathcal{L}) \xrightarrow{\Omega} \mathcal{S}_*\mathcal{O}(\Delta) \xrightarrow{Tr_{\mathfrak{S}}} \mathcal{O}_H.$$

Def^n: [Yang-Zhao] $Ell_\hbar(n_+) = \mathcal{SH}^{sph} \hookrightarrow \mathcal{O}_H$.

algebra object in $(Coh(H), \otimes_\hbar)$

§5: Double construction

Digression: Topologies

Recall: $K = \Gamma(E-P, \mathcal{O})$

$$\mathbb{O} = \prod_{x\in P} \hat{\mathcal{O}}_{E,x}$$

$$\mathbb{A} = \prod_{x\in P}{}' \operatorname{Frac}(\hat{\mathcal{O}}_{E,x})$$

Archetype:

$\mathbb{C}[z]$

$\mathbb{C}[[z]]$

$\mathbb{C}((z))$

These are ind-pro-finite dimensional vector spaces.

Tensor products: f.d.

$\underline{\otimes}$: $\operatorname{colim}_m \lim_n V_{m,n} \otimes \operatorname{colim}_p \lim_q W_{p,q}$

$$= \operatorname{colim}_p \operatorname{colim}_m \lim_q \lim_n (V_{m,n} \otimes W_{p,q})$$

$\vec{\otimes}$ (not symmetric):

$$\operatorname{colim}_m \lim_n V_{m,n} \vec{\otimes} \operatorname{colim}_p \lim_q W_{p,q}$$

$$= \operatorname{colim}_p \lim_q \operatorname{colim}_m \lim_n (V_{m,n} \otimes W_{p,q})$$

E.g. $\mathbb{C}((x)) \otimes \mathbb{C}((y)) = \mathbb{C}[[x,y]][x^{-1}, y^{-1}]$

$\mathbb{C}((x)) \vec{\otimes} \mathbb{C}((y)) = \mathbb{C}((x))((y))$ ← e.g. $\Rightarrow \frac{1}{x-y} = \sum_{r\geq}$

For X scheme, define

$$Shv(X) = \left\{ \begin{array}{l} \text{sheaves of ind-pro-f.d.v.s} \\ + \mathcal{O}_X\text{-action} \end{array} \right\}$$

$\rightsquigarrow$ Objects $K, \mathcal{O}, \mathbb{A} \in \mathrm{Shv}(E)$.

Define on $E^{(v)} \subset H$

$$K_H = \mathcal{S}_*(\underbrace{K \boxtimes \cdots \boxtimes K}_{\sum v^i \text{ times}})^{G_v}$$

$$\mathbb{A}_H = \mathcal{S}_*(\mathbb{A} \boxtimes \cdots \boxtimes \mathbb{A})^{G_v}$$

$$\mathcal{O}_H = \mathcal{S}_*(\mathcal{O} \boxtimes \cdots \boxtimes \mathcal{O})^{G_v}$$

and $\mathrm{Hom}(\mathbb{A}_H, \mathbb{C}) =: \mathbb{A}_H^* \twoheadrightarrow \mathcal{O}_H^*, K_H^*$

Consider $(\mathrm{Shv}(H^+ \times H^-), \otimes_\hbar)$ where

$$\mathcal{F} \otimes_\hbar \mathcal{G} = \mathcal{S}_*((\mathcal{L} \boxtimes \tau^* \mathcal{L}) \otimes (\mathcal{F} \boxtimes \mathcal{G}))$$

and $\vec{\otimes}_\hbar = \cdots \quad \vec{\boxtimes}$

<u>Subcategories</u>:

$$H^+ = H^+ \times \{0\} \longrightarrow H^+ \times H^-$$
$$H^- = \{0\} \times H^- \longrightarrow H^+ \times H^-$$
$$H \xrightarrow{\text{diag}} H^+ \times H^-$$

$\Rightarrow \mathrm{Shv}(H^\pm), \mathrm{Shv}(H) \longrightarrow \mathrm{Shv}(H^+ \times H^-)$. via push$\{$

$$\mathcal{E}ll_\hbar(n_\pm, \square) = \mathcal{E}ll_\hbar(n_\pm) \otimes_{\mathcal{O}_{H^\pm}} \square_{H^\pm}$$

The doubles are

(i) $\mathcal{E}ll_\hbar(\mathfrak{g}_{\mathbb{A}}) = \mathbb{A}_H^* \vec{\otimes}_\hbar (\mathcal{E}ll_\hbar(n_{+,\mathbb{A}}) \otimes_\hbar \mathcal{E}ll_\hbar(n_{-,\mathbb{A}})) \vec{\otimes}_\hbar \mathbb{A}_H^*$

(ii) $\mathcal{E}ll_\hbar(\mathfrak{g}_K) = (\mathcal{E}ll_\hbar(n_{+,K}) \otimes_\hbar \mathcal{E}ll_\hbar(n_{-,K})) \otimes_\hbar \mathcal{O}_H^*$

(iii) $\mathcal{E}ll_\hbar(\mathfrak{g}_{\mathcal{O}}) = K_H^* \otimes_\hbar (\mathcal{E}ll_\hbar(n_{+,\mathcal{O}}) \otimes_\hbar \mathcal{E}ll_\hbar(n_{-,\mathcal{O}}))$.

w/ Drinfeld double products.

<u>Example (Yangian)</u>: Can do this for $X = \mathbb{P}^1$, $\hbar \in$
$P =$

$\rightsquigarrow K = \mathbb{C}[z]$, $\mathcal{O} = \mathbb{C}[[z^{-1}]]$, $\mathbb{A} = \mathbb{C}((z^{-1}))$.

$\Gamma : \mathrm{Shv}(H^+ \times H^-) \longrightarrow \mathrm{IndProVect}^{fd}$ is monoi

$\Gamma(\mathcal{E}ll_\hbar(\mathfrak{g}_K)) = Y_\hbar(\mathfrak{n}_+) \otimes Y_\hbar(\mathfrak{n}_-) \otimes \mathrm{Sym}\left(\bigoplus_i z^{-1}\mathbb{C}[z\right.$

$z^{-1}\mathbb{C}[z] \cong \mathcal{O}^*$ pairing is $\mathrm{Res}_{z=\infty}(\ )(\ $

$$\Gamma(\mathcal{E}ll_\hbar(\mathfrak{g}_K)) \otimes_{\mathrm{Sym}(\bigoplus_i \mathbb{C}z^{-1})} \mathbb{C}_{z^{-1} \mapsto 1} = Y_\hbar(\mathfrak{g}).$$