Purity for G-Higgs bundles

partly joint work with Tasuki Kinjo

⓪ Introduction :

conjecture Halpern-Leistner

conjecture de dualité

① Reductive groups

② Cohomological integrality

③ Critical cohomological integrality

④ Dimensional reduction

⑤ Purity

⑥ Langlands duality conjecture for 3-manifolds

⑦ Langlands duality for the 3-torus.

⓪ Introduction

Bigger project: understand the geometry and topology of various stacks arising in algebraic geometry / representation theory.

ⓐ

0-shifted symplectic stacks [PTVV]
- * Higgs bundles / sm proj curve
- * representations of preprojective algebras
- * local systems over Riemann surfaces
- * vector bundles / symplectic surfaces

$\mathfrak{M}$ derived stack

$\mathbb{L}_{\mathfrak{M}}$ cotangent complex = complex of vector bundles $\left[\cdots \longrightarrow V^{i-1} \xrightarrow{d^{i-1}} V^{i} \xrightarrow{d^{i}} V^{i+1} \longrightarrow \cdots\right]$

$\mathbb{L}_{\mathfrak{M}} \cong \mathbb{L}_{\mathfrak{M}}^{\vee}$ induced by a symplectic form: $\omega \in H^0(\mathfrak{M}, \Lambda^2 \mathbb{L}_{\mathfrak{M}})$.

in particular: $\mathbb{L}_{\mathfrak{M}}$ is self-dual

Conjecture [Halpern-Leistner] - **Theorem** (H, 2024)

Let $\mathfrak{M} \xrightarrow{\pi} \mathcal{M}$ a 1-Artin derived stack $\mathfrak{M}$ admitting a good moduli space $\underbrace{\mathcal{M}}_{\text{algebraic space}}$.

If
- $\mathcal{M}$ is proper
- $\mathbb{L}_{\mathfrak{M}}$ is self-dual

then $H_*^{BM}(\mathfrak{M}, \mathbb{Q})$ carries a pure mixed Hodge structure.

$\Rightarrow$ positivity of certain enumerative invariants.

Strategy: work with local models of $\mathfrak{M} \xrightarrow{\pi} \mathcal{M}$ given by weak moment maps.

(b)

(-1)-shifted symplectic stacks $\left\{ \begin{array}{l} * \text{ Local systems on closed 3-manifolds} \\ \quad \mathcal{L}oc_G(M) = \{\pi_1(M) \to G\}/G. \\ * \text{ Coherent sheaves on CY 3-folds} \end{array} \right.$

$\mathbb{L}_{\mathfrak{M}} \cong \mathbb{L}^{\vee}_{\mathfrak{M}}[1]$ induced by a (-1)-shifted symplectic form

[Joyce et al] $\varphi_{\mathfrak{M}} \in \mathrm{Perv}(\mathfrak{M})$ a perverse sheaf.

Conjecture (Safronov) Let M be a closed 3-manifold. Then, there exists an isomorphism

$$H^*(\mathcal{L}oc_G(M), \underbrace{\varphi_{\mathcal{L}oc_G(M)}}_{\text{vanishing cycle cohomology}}) \cong H^*(\mathcal{L}oc_{G^L}(M), \varphi_{\mathcal{L}oc_{G^L}(M)})$$

Theorem (H-Kinjo) $M = T^3$ 3-torus $(\pi_1(M) \cong \mathbb{Z}^3)$

and $G/G^L = \left| \begin{array}{l} SL(n)/PGL(n) \\ SO(2n+1)/Sp(2n) \\ Spin(7)/PSp(6) \\ E_6^{sc}/E_6^{ad} \end{array} \right.$

Strategy: reduction to BPS cohomology + explicit formula for $\dim_{\mathbb{Q}} H^*_{BPS}(\mathcal{L}oc_G(M))$ - can be checked for any G/G^L in principle

Tool for both results: cohomological integrality

① **Reductive groups**

G reductive group
connected for simplicity / D. talk for non-connected ones.

$G = GL_n, SL_n, \mathbb{G}_m, \quad SO(2n+1), Sp(2n), \ldots$

$T \subset G$ max torus $\qquad (\mathbb{C}^*)^n \subset GL_n \ldots$

X smooth affine G-variety $\qquad G \times X \to X$ + usual axioms

For simplicity : $X = V$ linear representations of G:

V vector space + $G \to GL(V)$ group homomorphism

e.g. $\underset{\text{defining}}{GL_2 \curvearrowright \mathbb{C}^2}$; $\underset{\text{conjugation}}{GL_2 \curvearrowright Mat_{2\times 2}(n)}$

sufficient : Luna étale slice theorem.

$$\underset{\text{characters}}{X^*(T)} \times \underset{\text{cocharacters}}{X_*(T)} \to \mathbb{Z}$$

$$(\alpha, \lambda) \mapsto \alpha \circ \lambda \in End(\mathbb{G}_m) \cong \mathbb{Z}.$$

Weights $\quad G \curvearrowright V \cong \bigoplus_{\alpha \in X^*(T)} V_\alpha \qquad V_\alpha = \{ v \in V \mid t \cdot v = \alpha(t) \cdot v \quad \forall t \in T \}$

$$W(V) = \{ \alpha \in X^*(T) \mid V_\alpha \neq 0 \}.$$

Self-duality $\quad V \cong V^*$ as G-representations
given by $V \times V \xrightarrow{b} \mathbb{C}$ bilinear

orthogonal $\quad b$ can be chosen orthogonal $\to$ induced a quadratic function on V.

symplectic $\quad$ " $\quad$ " $\quad$ symplectic

non-example: $GL_2 \curvearrowright \mathbb{C}^2 = \underset{(1,0)}{\mathbb{C}e_1} \oplus \underset{(0,1)}{\mathbb{C}e_2}$

ex · * any representation of SL_2
* T^*V for any rep. of G
* adjoint representation of G

$\Big\}$ orthogonal

* $Sp(2n) \curvearrowright \mathbb{C}^{2n}$ symplectic and <u>not orthogonal</u>

Torus: V self-dual $\Leftrightarrow$ $\dim V_\alpha = \dim V_{-\alpha}$ $\forall \alpha \in X^*(T)$.

Weyl group $W = N_G(T)/T$

$$N_G(T) = \{ g \in G \mid gTg^{-1} = T \}$$

$$W_{GL_n} \cong W_{SL_n} \cong \mathfrak{S}_n \quad \text{symmetric group.}$$

In general: W is a Coxeter group.

T torus: $W_T = \{e\}$

$W \curvearrowright$ weights of V $= \mathcal{W}(V)$.

Induction morphisms

Cohomological Parabolic induction

V representation of G

$\lambda : \mathbb{G}_m \to T$ cocharacter

$G^\lambda = \{ g \in G \mid \lambda(t) g \lambda(t)^{-1} = g \quad \forall t \in \mathbb{C}^* \}$

$\subset G$ levi subgroup

Note G^λ reductive

$T \subset G^\lambda$.

$V^\lambda = \{ v \in V \mid \lambda(t) v \lambda(t)^{-1} = v \quad \forall t \in \mathbb{C}^* \}$

$\subset V$ subspace

$G^{\lambda \geq 0} = \{ g \in G \mid \lim_{t \to 0} \lambda(t) g \lambda(t)^{-1} \text{ exists} \}$

$\subset G$ parabolic subgroup

$V^{\lambda \geq 0} = \{ v \in V \mid \lim_{t \to 0} \lambda(t) \cdot v \text{ exists} \}$

$\subset V$

subspace

$G = Gl_n \qquad V = T^* \mathbb{C}^n$

$\mathbb{G}_m \longrightarrow Gl_n$

$t \longmapsto \begin{pmatrix} t^2 & & 0 \\ & t & \\ 0 & & 1 \end{pmatrix}$

$\begin{pmatrix} \boxed{*} & & 0 \\ & \boxed{*} & \\ 0 & & \boxed{*} \end{pmatrix}$

$V^\lambda = T^* \begin{pmatrix} 0 \\ 0 \\ * \end{pmatrix}$

$\begin{pmatrix} * \\ 0 \end{pmatrix}$

$V^{\lambda \geq 0} = \mathbb{C}^n \oplus \begin{pmatrix} 0 \\ 0 \\ * \end{pmatrix}$

Induction diagram

$$\begin{tikzcd}
 & V^{\lambda \geq 0}/G^{\lambda \geq 0} \arrow[dl, "q_\lambda"', "\text{smooth}"] \arrow[dr, "p_\lambda", "\text{proper}"'] & \\
V^\lambda / G^\lambda & & V/G
\end{tikzcd}$$

$$\mathrm{Ind}_\lambda := p_{\lambda *} q_\lambda^* : H^*(V^\lambda/G^\lambda) \longrightarrow H^*(V/G)$$

cohomological parabolic induction

$$\mathrm{Ind}_\lambda : \mathbb{Q}[x_1, \ldots, x_r]^{W^\lambda} \longrightarrow \mathbb{Q}[x_1, \ldots, x_r]^W.$$

$\exists$ translation of coh degree making Ind_λ graded.

Explicit formula:

$\alpha \in X^*(T)$ may be seen as an element of $H_T^*(pt) \cong \mathrm{Sym}(\mathfrak{t}^*)$

$\alpha : T \to \mathbb{G}_m \quad d\alpha(1) : \mathfrak{t} \to \mathbb{C} \in \mathfrak{t}^*$.

$$k_\lambda := \frac{\prod_{\substack{\alpha \in \mathcal{W}(V) \\ \langle \lambda, \alpha \rangle < 0}} \alpha^{\dim V_\alpha}}{\prod_{\substack{\alpha \in \mathcal{W}(\mathfrak{g}) \\ \langle \lambda, \alpha \rangle < 0}} \alpha^{\dim \mathfrak{g}_\alpha}} \in \mathrm{Frac}\left(H_T^*(pt)\right)$$

$$\mathrm{Ind}_\lambda(f) = \frac{1}{|W^\lambda|} \sum_{w \in W} w \cdot (f k_\lambda).$$

Proof: Calculation after localization and computation of Euler class using Borel-Weil-Bott thm.

Refinement of this approach

$$\begin{tikzcd}
 & V^{\lambda\geq 0}/G^{\lambda\geq 0} \arrow[dl, "q_\lambda"'] \arrow[dr, "p_\lambda"] & \\
V^\lambda/G^\lambda \arrow[d, "\pi_\lambda"'] & \circlearrowright & V/G \arrow[d, "\pi"] \\
V^\lambda /\!/ G^\lambda \arrow[rr, "\iota_\lambda"'] & & V/\!/G
\end{tikzcd}$$

finite morphism

$$\mathrm{Ind}_\lambda : \iota_{\lambda *} \pi_{\lambda *} \mathbb{Q}^{\mathrm{vir}}_{V^\lambda/G^\lambda} \longrightarrow \pi_* \mathbb{Q}^{\mathrm{vir}}_{V/G}[\dim V^{\lambda>0} - \dim V^{\lambda<0}]$$

morphism in $\mathcal{D}_c(V/\!/G)$

$\mathcal{D}(\mathrm{MMHM}(V/\!/G))$

Cohomological integrality theorem

$$\lambda \sim \mu \iff \begin{cases} G^\lambda = G^\mu \\ V^\lambda = V^\mu \end{cases}$$

$\rightsquigarrow \mathcal{P}_V = X_*(T)/\sim$ finite set

$\circlearrowleft$

W

$G_\lambda = \ker(G^\lambda \longrightarrow GL(V^\lambda)) \cap Z(G^\lambda) \subset G$ normal subgroup

$W_\lambda = \{w \in W \mid w\cdot\lambda \sim \lambda\} \subset W$ subgroup

$\varepsilon_{V,\lambda} : W_\lambda \longrightarrow \{\pm 1\}$ such that

$k_{w\cdot\lambda} = \varepsilon_{V,\lambda}(w) k_\lambda \quad \forall w \in W_\lambda$. [requires V self-dual]

Theorem (H-2024) Let V be a weakly symmetric representation of a reductive group G.

There exists bounded cohomologically graded complexes of monodromic mixed Hodge modules $\mathcal{BPS}_{V/G,\lambda}$, which are W_λ-equivariant, such that the morphism

$$\bigoplus_{\tilde{\lambda}\in \mathcal{P}_V/W} \left[(\iota_\lambda)_* \mathcal{BPS}_{V/G,\lambda} \otimes H^*(pt/G_\lambda) \right]^{\varepsilon_{V,\lambda}} \xrightarrow{\mathrm{Ind}_\lambda} \pi_* \mathbb{Q}^{vir}_{V/G}$$

is an isomorphism in $\mathcal{D}^+(\mathrm{MMHM}(V/\!/G))$.

Conjecture – **Theorem** when V is orthogonal [Bu-Davison-Ibáñez Núñez-Kinjo-Pădurariu]

$$\mathcal{BPS}_{V/G,\lambda} \simeq \begin{cases} \mathcal{IC}(V^\lambda/\!/G^\lambda) \otimes \mathcal{L}^{\dim G_\lambda/2} & \text{if } \dim V^\lambda/G^\lambda = \dim V^\lambda/\!/G^\lambda - \dim G_\lambda \\ 0 & \text{otherwise} \end{cases}$$

Direct calculations when $G = \mathbb{C}^*$, $G = Sp(2n)$

"cuspidal cohomology is intersection cohomology".

→ Interplay between $H^*_G(V)$ and $IH^*(V/\!/G)$
easy difficult.

Examples: cohomological integrality isomorphisms.

① $\mathbb{C}^* \curvearrowright V = \bigoplus_{k\in\mathbb{Z}} V_k$

$P_{d_0} = \mathbb{Q}[x]_{\deg < \sum_{k>0} \dim V_k}$

$P_{d_1} = \mathbb{Q}$.

Integrality isomorphism

$$\begin{array}{ccc} P_{d_0} \oplus (P_{d_1} \otimes \mathbb{Q}[x]) & \longrightarrow & \mathbb{Q}[x] \\ (f, g) & \longmapsto & f + k_{d_1} \cdot g. \end{array}$$

clearly an isomorphism

---

② $\overset{G}{\overset{\parallel}{GL_2(\mathbb{C})}} \curvearrowright \overset{V}{\overset{\parallel}{(T^*\mathbb{C}^2)^g}} \quad g \geq 0 \qquad T = (\mathbb{C}^*)^2 \subset GL_2(\mathbb{C})$

Some calculations:

$$P_{d_0} = \bigoplus_{j=0}^{g-2} \mathbb{Q}(x_1+x_2)^j \subset H^*(V/G) \cong \mathbb{Q}[x_1+x_2, x_1x_2]$$

$$P_{d_1} = \bigoplus_{j=0}^{g-1} \mathbb{Q}x_2^j \subset H^*(V^{d_1}/G^{d_1}) \cong \mathbb{Q}[x_1, x_2]$$

$$P_{d_2} = \mathbb{Q} \subset H^*(V^{d_2}/G^{d_2}) \cong \mathbb{Q}[x_1, x_2]$$

$$P_{d_3} = \{0\} \subset \mathbb{Q}[x_1 + x_2, x_1 x_2].$$

Integrality isomorphism

$$P_{d_0} \oplus (P_{d_1} \otimes \mathbb{Q}[x_1]) \oplus (P_{d_2} \otimes \mathbb{Q}[x_1, x_2])^{sgn} \longrightarrow \mathbb{Q}[x_1 + x_2, x_1 x_2]$$

$$(f, h, k) \mapsto f + \frac{x_1^g h(x_1, x_2) - x_2^g h(x_2, x_1)}{x_1 - x_2} + 2(x_1 x_2)^g \frac{k(x_1, x_2)}{x_1 - x_2}.$$

(3) $Sp(2n) \curvearrowright \mathbb{C}^{2n}$

$P_d = \mathbb{Q}$ for d generic (not on any wall of the hyperplane arrangement)

$$W_{Sp(2n)} \cong (\mathbb{Z}/2)^n \rtimes \mathfrak{S}_n$$

$$H^*_{Sp(2n)}(pt) \cong \mathbb{Q}[x_1^2, \dots, x_n^2]^{\mathfrak{S}_n}$$

$sgn : W_{Sp(2n)} \to \{\pm 1\}$ sign character, sends any reflection to -1.

$$\mathbb{Q}[x_1, \dots, x_n]^{\varepsilon} \xrightarrow{\sim} \mathbb{Q}[x_1^2, \dots, x_n^2]^{\mathfrak{S}_n}$$

$(-1)^n \prod x_i$

$$f \longmapsto \sum_{w \in W} w \cdot \left(f \frac{\quad}{\prod_{j>i}(x_i - x_j) \prod_{j>i}(x_i + x_j) \cancel{\prod_i x_i}} \right)$$

4 Calculation of pushforward [using [BDINKP]]

$\mathbb{C}^{2n+1} \circlearrowleft SO(2n+1)$

$$\begin{array}{ccc} \mathbb{C}^{2n+1}/SO(2n+1) & \qquad & \pi_* \mathcal{IC}(\mathbb{C}^{2n+1}/SO(2n+1)) \\ \big\downarrow \pi & & \cong \\ \mathbb{C}^{2n+1}/\!/SO(2n+1) \cong \mathbb{A}^1 & & \mathcal{IC}(\mathbb{A}^1)[-n] \otimes H_T(pt)^{\varepsilon} \\ & & \oplus\ \mathcal{IC}(\mathcal{L}) \otimes H_T(pt)^{sgn} \end{array}$$

where $\varepsilon : W_{SO(2n+1)} \cong (\mathbb{Z}/2)^n \rtimes \mathfrak{S}_n \longrightarrow \{\pm 1\}$ is the product of sgn and $\varepsilon' : W_{SO(2n+1)} \to (\mathbb{Z}/2)^n \to \{\pm 1\}$ sending each generator of $(\mathbb{Z}/2)^n$ to -1.

5 $SL_2 \circlearrowright \mathrm{Mat}_{2\times n}$

$P_{triv} = \mathbb{Q}[x^2]_{\deg \leq 2(n-2)}$ $\qquad \deg x = 2$

$\mathrm{Mat}_{2\times n} /\!/ SL_2$ is the affine cone over $Gr(2,n)$.

$\mathrm{IH}^*(\mathrm{Mat}_{2\times n} /\!/ SL_2)$ = primitive cohomology of $Gr(2,n)$.

$$\mathrm{IH}^j(\mathrm{Mat}_{2\times n} /\!/ SL_2) = \begin{cases} \mathbb{Q} & \text{if } j \equiv 0 \ (4) \ \& \ j \leq 2(n-2) \\ 0 & \text{otherwise.} \end{cases}$$

Critical cohomological integrality

Corollary In the same situation as before, for any function $f : X \to \mathbb{C}$,

$$\pi_* \varphi_f \mathbb{Q}^{vir}_{X/G} \cong \bigoplus_{\tilde{\lambda} \in \mathcal{P}_X / W} \left[\iota_{\lambda *} \varphi_f \mathcal{BPS}_{X/G, \lambda} \otimes H^*(pt/G_\lambda) \right]^{W_\lambda}$$

Why doing that? $\to$ dimensional reduction (Kontsevich-Soibelman, Davison, Kinjo)

Y smooth alg variety $/\mathbb{C}$

$$\begin{array}{c} E \\ \downarrow \\ Y \end{array}$$ vector bundle $\quad Y \xrightarrow{s} E^\vee$ section $\quad$ produces $f : E \to \mathbb{A}^1$ regular function.

$$Z = s^{-1}(0)$$

$$\begin{tikzcd} \bar{Z} = \pi^{-1}(Z) \arrow[r, hook, "\bar{\iota}"] \arrow[d, "\iota'"'] & E \\ E_0 = f^{-1}(0) \arrow[r, hook, "\iota"'] & E \end{tikzcd}$$

The adjunction morphism $id \longrightarrow \bar{\iota}_* \bar{\iota}^*$ gives an isomorphism

$$\pi_! \varphi_f \pi^* \to \pi_! \bar{\iota}_* \bar{\iota}^* \pi^* \cong \pi_! \underbrace{\varphi_f}_{\text{vanishing cycle sheaf functor}} (id \longrightarrow \bar{\iota}_* \bar{\iota}^*) \pi^* : \mathcal{D}_c(X) \longrightarrow \mathcal{D}_c(X)$$

$$\Rightarrow \boxed{\pi_* \varphi_f \mathbb{Q}_E^{vir} \cong \mathbb{D}\mathbb{Q}^{vir}_{s^{-1}(0)}} \Rightarrow \underbrace{H^*(E, \varphi_f \mathbb{Q}_E^{vir})}_{\text{vanishing cycle cohomology}} \cong \underbrace{H^{BM}_{-*-2vir}(s^{-1}(0), \mathbb{Q})}_{\text{Borel–Moore homology}}$$

Local description of self-dual stacks

X weakly symplectic affine algebraic variety

$$TX \underset{\varphi}{\cong} T^*X$$

$G \circlearrowright X$ weakly Hamiltonian action:

$\exists \quad \mu : X \to \mathfrak{g}^*$ such that

$$\begin{tikzcd} X \times \mathfrak{g} \arrow[r, "d\mu"] \arrow[d, "\varphi"'] & T^*X \arrow[d, "\psi"] \\ X \times \mathfrak{g} \arrow[r, "a"] & TX \end{tikzcd}$$

diagram of vector bundles over X.

commutes over $\mu^{-1}(0) \subset X$.
subscheme.

+ can assume $\mu(x)(\xi) = 0$ for $\xi \in \mathrm{Lie}(\ker(G \to \mathrm{Aut}(X)))$

Consider $\mu^{-1}(0)/G \xrightarrow{\pi} \mu^{-1}(0)/\!/G$, related to quiver varieties studied by Nakajima

$f: X\times \mathfrak{g} \longrightarrow \mathbb{C}$
$(x, \xi) \longmapsto \mu(x)(\xi)$.

Dimensional reduction $p: X\times\mathfrak{g}/G \to X/G$; $p: X\times\mathfrak{g}/\!/G \to X/\!/G$

$$p_* \varphi_f \mathbb{Q}^{vir}_{X\times\mathfrak{g}/G} \cong \mathbb{D}\mathbb{Q}^{vir}_{\mu^{-1}(0)/G}.$$

Corollary: The critical coh. integrality isomorphism provides an isomorphism

$$\pi_* \mathbb{D}\mathbb{Q}^{vir}_{\mu^{-1}(0)/G} \cong \bigoplus_{\tilde{\lambda}\in P_X/W} \left[p_* q_{\lambda *} \varphi_f \mathcal{BPS}_{V/G,\lambda} \otimes H^*(pt/G_\lambda)\right]^{W_\lambda}.$$

where $q_\lambda : V^\lambda/\!/G^\lambda \longrightarrow V/\!/G$.

Corollary [purity] $X \circlearrowleft G$ weakly Hamiltonian

① $\pi_* \mathbb{D}\mathbb{Q}^{vir}_{\mu^{-1}(0)/G} \in \mathcal{D}^+_G(\mathrm{MMHM}(\mu^{-1}(0)))$ is a pure complex.

② $T^*V \circlearrowleft G$ Hamiltonian action on T^*V; $\mu: T^*V \to \mathfrak{g}^*$ moment map.

$H^{BM}_*(\mu^{-1}(0)/G, \mathbb{Q})$ is pure.

Then $H^{BM}_*(\mu^{-1}(0)/G, \mathbb{Q})$ carries a pure MHS

③. If $\mathfrak{X}$ is a 1-Artin derived stack with affine diagonal, good moduli space $\mathfrak{X} \xrightarrow{\pi} X$ and X is proper, $H_*^{BM}(\mathfrak{X}, \mathbb{Q})$ has pure MHS.
[conjecture of Halpern-Leistner]

④ $H_*^{BM}(\mathcal{H}iggs_G^{ss}(C), \mathbb{Q})$ carries a pure mixed Hodge structure.

Langlands duality for 3-torus

Donaldson-Thomas perverse sheaf

$\mathfrak{M}$ a (-1)-shifted symplectic stack

Darboux theorem: $\mathfrak{M}$ is étale locally isomorphic to a critical locus:

$\mathfrak{X}$ smooth Artin stack

$f : \mathfrak{X} \to \mathbb{A}^1$ regular function

$$\begin{tikzcd} \mathrm{crit}(f) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathfrak{X} \arrow[d, "df"] \\ \mathfrak{X} \arrow[r, "0"'] & T^*\mathfrak{X} \end{tikzcd}$$

On $\mathrm{crit}(f)$, • $\varphi_f \in \mathrm{Perv}(\mathrm{crit}(f))$ vanishing cycle sheaf

• Glue to $\varphi_{\mathfrak{M}} \in \mathrm{Perv}(\mathfrak{M})$. "DT-perverse sheaf"

M a 3-manifold

[PTVV] $\mathcal{L}oc_G(M)$ is (-1)-shifted symplectic

$\rightsquigarrow \varphi_{\mathcal{L}oc_G(M)} \in \mathrm{Perv}(\mathcal{L}oc_G(M))$

BPS sheaf: $\pi : \mathcal{L}oc_G(M) \to \mathrm{Loc}_G(M)$ good moduli space

$$\mathcal{BPS}_{\mathrm{Loc}_G(M)} := {}^p\mathcal{H}^{\dim Z(G)}\left(\pi_* \varphi_{\mathcal{L}oc_G(M)}\right) \in \mathrm{Perv}(\mathrm{Loc}_G(M))$$

BPS cohomology $H^*_{BPS}(\mathrm{Loc}_G(M)) := H^*\left(\mathrm{Loc}_G(M), \mathcal{BPS}_{\mathrm{Loc}_G(M)}\right).$

$M = T^3$

$$\mathcal{L}oc_G(T^3) \simeq \overbrace{\{(g_1, g_2, g_3) \in G \mid [g_i, g_j] = 1\}}^{\mathcal{C}^{(3)}(G) \text{ triple commuting variety}} / G$$

$$\begin{array}{c} \mathcal{L}oc_G(T^3) \\ \downarrow \pi \\ \mathrm{Loc}_G(T^3) \cong \mathcal{C}^{(3)}(G) /\!/ G \end{array}$$

Theorem (H- Kinjo) Assume G is semisimple

$$\mathcal{BPS}_{\mathcal{L}oc_G(M)} \cong \bigoplus_{\substack{g := (g_1, g_2, g_3) \in \mathcal{C}^{(3)}(G)/\!/G \\ \text{isolated triple}}} \mathbb{Q}_{\{g\}}^{\oplus r_g}$$

where r_g = # distinguished nilpotent orbits of $C_G(g)$.

Proof: * $g \in \mathrm{Loc}_G(T^3)$ closed point.

Étale neighbourhood is given by

$$\mathcal{C}^{(3)}(\mathrm{Lie}\ C_G(g)) /\!/ C_G(g)$$

* Computation of $\mathcal{BPS}_{\mathcal{C}^{(3)}(\mathfrak{g})/\!/G}$ for reductive groups G:

$$\dim H^0_{BPS}(\mathcal{C}^{(3)}(\mathfrak{g})/\!/G) = \#\text{ distinguished nilpotent orbits of } G.$$

Corollary: $\dim H^0_{BPS}\left(\mathrm{Loc}_{SL_n}(T^3)\right) = n^3 = \dim H^0_{BPS}\left(\mathrm{Loc}_{PGL_n}(T^3)\right)$

$$\dim H^0_{BPS}\left(\mathrm{Loc}_{SO(2n+1)}(T^3)\right) = \frac{1}{8}\left[x^{2n+1}\right]\prod_{k\geq 0}\left(1+x^{2n+1}\right)^8$$

$$\|$$

$$\dim H^0_{BPS}\left(\mathrm{Loc}_{Sp(2n)}(T^3)\right) = \left[x^n\right]\prod_{k\geq 1}\left(1+x^n\right)^8$$

identity involving Jacobi θ-functions.

$$\dim H^0_{BPS}\left(\mathrm{Loc}_{E_6^{ad/sc}}(T^3)\right) = 416.$$