GKM – Homology of aff Spr fibers in the unramified case

$\gamma \in \mathfrak{t}(\mathcal{O})$ rss

/ Parity of equivalued ASF.

Equivariant Cohomology of affine Springer fibers

blue = remarks for myself

I blatantly refer to my talk at the SERG Reading group for basics on equivariant cohomology

G cpx reductive group.

$F = \mathbb{C}((\varepsilon))$

$\mathcal{O} = \mathbb{C}[[\varepsilon]]$

$Gr_G = G(F)/G(\mathcal{O})$

$\gamma \in \mathfrak{t}(\mathcal{O})$ regular semisimple. ["unramified case"]

affine Springer fiber

$$X_\gamma = \{ x \in Gr_G \mid \mathrm{Ad}(x^{-1})\gamma \in \mathfrak{g}(\mathcal{O}) \}.$$

finite dimensional ind-variety.

$T(\mathbb{C}) \curvearrowright X_\gamma$: $\quad z \in T(\mathbb{C}) \quad z \cdot g\,G(\mathcal{O}) = (\mathrm{Ad}(z)\,g)\,G(\mathcal{O})$.

Question : $H_*^{T(\mathbb{C})}(X_\gamma) = ?$

Rk: $\gamma = a t^d \in \mathfrak{t}(F)$ $\quad a \in \mathfrak{t}^{rss}$

$d \geq 0$

$$\lim_{d \to \infty} X_\gamma = X$$

Answer to the question + limit gives $H_*^{T(\mathbb{C})}(X)$.

$\Lambda \subset T(F)$ lattice of translations

$$\begin{tikzcd}
1 \arrow[r] & T(\mathcal{O}) \arrow[r] & T(F) \arrow[r, "val"] & X_*(T) \arrow[r] \arrow[l, bend left] & 1
\end{tikzcd}$$

splitting (choosing uniformizing parameter ε for $\mathbb{C}((\varepsilon))$)

w/ image Λ

$G \supset B \supset T \rightsquigarrow \Phi^+$ positive roots

$\alpha \in \Phi^+$

$\alpha^\vee : F^\times \longrightarrow T(F) \rightsquigarrow \alpha^\vee \in \Lambda$

$\alpha^\vee \in t = t^{**} \rightsquigarrow$ degree 1-monomial $x_{\alpha^\vee} \in S(t) = \mathrm{Sym}(t)$

$\alpha \in t^* \simeq X^*(T) \otimes_{\mathbb{Z}} \mathbb{C} \rightsquigarrow$ degree 1-differential operator $\partial_\alpha \in \mathcal{D}(t) = \mathrm{Sym}(t^*)$

$$\boxed{\partial_\alpha (x_{\alpha^\vee}) = 2}$$

$S(t)\{\partial_\alpha^d\}$ submodule of pol funct. annihilated by ∂_α^d.

$$L_{\alpha,\gamma} = \sum_{d=1}^{val(\alpha(\gamma))} (1-\alpha^\vee)^d \, \mathbb{C}[\Lambda] \otimes S(t)\{\partial_\alpha^d\}$$

Thm: Suppose $H_*(X_\gamma; \mathbb{C})$ is pure. Then the inclusion $\Lambda \subset X_\gamma$ induces an exact sequence

$$0 \to \sum_{\alpha\in\Phi^+} L_{\alpha,\gamma} \to H_*^{T(\mathbb{C})}(\Lambda) \to H_*^{T(\mathbb{C})}(X_\gamma) \to 0$$

where $\Lambda \cong T(F)/T(\mathcal{O})$ and $H_*^{T(\mathbb{C})}(\Lambda) = \mathbb{C}[\Lambda] \otimes S(t)$.

Fact GKM - Purity of equivalued Springer fibers

purity holds for $\gamma = a t^d$, $a \in t^{rss}$
$d \geq 0$.

define affine paving of X_γ by intersecting it w/ Schubert cells.

Limit

$$0 \to \sum_{\alpha \in \phi^+} \sum_{d=1}^{\infty} (1-\alpha^\vee) \mathbb{C}[\Lambda] \otimes S(t) \{ \partial_\alpha^d \} \to H_*^{T(\mathbb{C})}(\Lambda) \to H_*^{T(\mathbb{C})}(X) \to 0$$

$\|$

L_∞ is preserved by $\tilde{W} \rtimes \mathrm{Aut}$.

So $\tilde{W} \rtimes \mathrm{Aut}$ acts on $H_*^{T(\mathbb{C})}(X)$.

A subgroup action on the homology of affine Springer fibers:

$$(\tilde{W} \rtimes \mathrm{Aut})_\gamma = \{ \tau \in \tilde{W} \rtimes \mathrm{Aut} : \mathrm{val}(\tau\alpha(\gamma)) = \mathrm{val}(\alpha(\gamma)) \ \forall \alpha \in \phi^+ \}.$$

preserves the module of relations $\sum_{\alpha \in \phi^+} L_{\alpha,\gamma}$.

Homology of affine Springer fibers in the affine flag manifold.

$I \subset G(F)$ Iwahori

$$\begin{tikzcd} I \arrow[r] \arrow[d, hook] & B \arrow[d] \\ G(\mathcal{O}) \arrow[r] & G \end{tikzcd}$$

$Y = Y^G = G(F)/I$ affine flag manifold

$\Lambda \subset T(F)$ lattice of translations.

$\alpha^\vee \in \Phi^\vee(G,T) \rightsquigarrow \alpha^\vee(\varepsilon) \in \Lambda$.

$\tilde{W} = \Lambda \rtimes W$

$\alpha \in \Phi^+$ give $\partial_\alpha \in \mathcal{D}(t)$ differential operator.

$\alpha^\vee \in \Lambda$ coroot

$w_\alpha \in W$ reflection

Define
$$M_{\alpha,\gamma} = \sum_{d=1}^{\mathrm{val}(\alpha'(\gamma))} (1-\alpha^\vee)^d\, \mathbb{C}[\tilde{W}] \otimes_{\mathbb{C}} S(t)\{\partial_\alpha^d\}$$
$$+ \sum_{d=1}^{\mathrm{val}(\alpha'(\gamma))} (1-\alpha^\vee)^{d-1} (1-w_\alpha)\, \mathbb{C}[\tilde{W}] \otimes_{\mathbb{C}} S(t)\{\partial_\alpha^d\}.$$

$\gamma \in t(F)$

If G simply connected, Y^G classifies Iwahori subalgebras of $\mathfrak{g}(F)$

Y_γ those containing γ.

Thm: Let $\gamma \in \mathfrak{t}(G)$ be a regular element.

Suppose $H_*(Y_\gamma;\mathbb{C})$ is pure (implying formality)

Then the inclusion $\tilde{W} \subset Y_\gamma$ induces an exact sequence of $\mathcal{D}(t)$-modules:

$$0 \to \sum_{\alpha \in \phi^+} M_{\alpha,\gamma} \to \mathbb{C}[\tilde{W}] \underset{\mathbb{C}}{\otimes} S(t) \to H_*^{T(\mathbb{C})}(Y_\gamma) \to 0$$

$(\tilde{W} \rtimes \mathrm{Aut})_\gamma \curvearrowright$ on $H_*^{T(\mathbb{C})}(Y_\gamma)$; it restricts to an action on the ordinary homology

$$H_*(Y_\gamma) = H_*^{T(\mathbb{C})}(Y_\gamma)\{\mathcal{I}\},$$ subgroup of $H_*^{T(\mathbb{C})}(Y_\gamma)$ annihilated by the augmentation ideal $\mathcal{J} \subset \mathcal{D}$.

Aut = automorphism group of the based root datum for G

$\subset$ Dynkin diagram automorphisms of G

$=$ in simply connected and adjoint cases

$X_*(T) = \mathbb{Z}\phi^\vee$ $\qquad$ $X^*(T) = \mathbb{Z}\phi$

$\tilde{W} = \Lambda \rtimes W$ extended affine Weyl group

$\tilde{W} \rtimes \mathrm{Aut}$ $\circlearrowright$ on $T(\mathbb{C})$ and on the root system $\Phi(G,T)$.

through $W \rtimes \mathrm{Aut}$

$\circlearrowright$ on Λ

It acts diagonally on the left on

$$H_*^{T(\mathbb{C})}(\Lambda) \cong \mathbb{C}[\Lambda] \otimes_{\mathbb{C}} S(t)$$

$(\tilde{W} \rtimes \mathrm{Aut})_\gamma$ passes acts on $H_*^{T(\mathbb{C})}(Y_\gamma)$

and on $H_*(Y_\gamma) = H_*^{T(\mathbb{C})}(Y_\gamma)\{\mathcal{J}\}$

$\hookrightarrow$ aug. ideal of $\mathcal{D}(t)$.

Springer action (Minh-Tâm talk)

regular $\otimes$ trivial $\circlearrowright$ $\tilde{W}$ from the right on

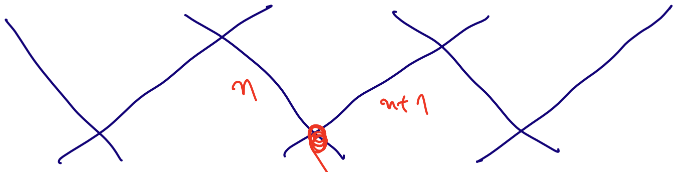
$\mathbb{C}[\tilde{W}] \otimes S(t)$.

It preserves each $M_{\alpha,\gamma}$ $\alpha \in \Phi^+$

Passes to an action on $H_*^{T(\mathbb{C})}(Y_\gamma)$ comp. w/ $\mathcal{D}(t)$-module structure & commute w/ $(\tilde{W} \rtimes \mathrm{Aut})_\gamma$ - action.

Coincides w/ Lusztig's Springer action.

Example: $G = SL_2$, $\gamma = t\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ equivalued Springer fiber

$X_\gamma =$ 

infinite chain of $\mathbb{P}^1$'s.

torus $T \simeq \mathbb{C}^* \subset G$

$\Lambda \subset \mathbb{C}((t))$, $\gamma\Lambda \subset \Lambda$.

$n \in \mathbb{Z}$

$$C_n = \Big\{ \Lambda \subset \mathbb{C}((t))^{\oplus 2} \ \Big|\ \underbrace{t^n\mathbb{C}[[t]] \oplus t^{-n+1}\mathbb{C}[[t]]}_{\Lambda'} \subset \Lambda \subset \underbrace{t^{n-1}\mathbb{C}[[t]] \oplus t^{-n}\mathbb{C}[[t]]}_{\frac{1}{t}\Lambda'} \Big\}$$

Then $C_n \simeq \mathbb{P}^1$

T-action

$$\tilde{u} = \begin{pmatrix} u & 0 \\ 0 & u^{-1}\end{pmatrix} \qquad g = \begin{pmatrix} 1 & x \\ 0 & 1\end{pmatrix}\begin{pmatrix} t^n & 0 \\ 0 & t^{-n}\end{pmatrix} \qquad x \in t^{2n-1}O_F / t^{2n}O_F$$

$$\tilde{u}\, g\, \tilde{u}^{-1} = \begin{pmatrix} 1 & u^2 x \\ 0 & 1\end{pmatrix}\begin{pmatrix} t^n & 0 \\ 0 & t^{-n}\end{pmatrix}$$

so $\mathbb{C}^*$ acts with weight 2 on each $\mathbb{P}^1$ component of X_γ with fixed points the intersection points $= \Lambda = \mathbb{Z}$

$$\underset{\cong}{\mathbb{C}[\Lambda]} \otimes \underbrace{H_*^{T(\mathbb{C})}(pt)}_{\shortparallel\ \mathbb{C}[x]}$$

$$H_*^{T(\mathbb{C})}(\mathbb{Z}) \longrightarrow H_*^{T(\mathbb{C})}(X_\gamma) \longrightarrow 0$$

- $\alpha \in \Phi^+$ unique positive root. $\quad \mathrm{val}(\gamma) = 1 \in \mathbb{Z}$
- $L_{\alpha,\gamma} = (1-\alpha^\vee)\,\mathbb{C}[\Lambda] \otimes \boxed{\mathbb{C}[x]\{\partial_\alpha\}}$

$\mathbb{C}[x]\{\partial_\alpha\}$, with $\partial_\alpha = \partial$: $\;=$ pol functions annihilated by ∂ $\;\cong \mathbb{C}$ constant polynomial functions.

So that $$\boxed{H_*^{T(\mathbb{C})}(X_\gamma) \simeq \mathbb{C}[\Lambda]\otimes\mathbb{C}[x] \Big/ (1-\alpha^\vee)\,\mathbb{C}[\Lambda]}$$

(1) Purity

X cpx proj variety

Deligne: increasing weight filtration (W^m) on $H^*(X;\mathbb{C})$

$\Rightarrow$ increasing weight filtration on $H_*(X,\mathbb{C})$ by duality

"X pure" means by definition

$$Gr^m W(H^i(X)) = 0 \text{ if } m \neq i$$

$$\Leftrightarrow \quad Gr^{-m} W(H_i(X)) = 0 \text{ if } m \neq i.$$

The strictness: If $f: X \to Y$ morphism of projective varieties, $f_*: H_*(X) \to H_*(Y)$ is strict w.r.t. the weight filtration:

$$(\mathrm{im}\, f_*) \cap W^m(H^*(Y)) = f_* W^m(H^*(X)).$$

allows to extend the definition of the weight filtration to any ind projective variety (inductive limit of projective varieties) $X = \varinjlim_{i \in I} X_i$ by setting

$$W^m H_*(X) = \varinjlim W^m(H_*(X_i))$$

$$\subset \varinjlim H_*(X_i) =: H_*(X).$$

[Homology commutes w/ direct limits]

[Rk : strictness of differential]

Affine Springer fibers are ind-varieties

$\Rightarrow$ homology/cohomology spaces can be infinite dimensional.

It's good to see them as module over an as big as possible algebra to get a good grasp on them.

[even better if we know (a part of) the structure of the category of modules over this algebra.

Here : Cherednik algebras . Subject of talk 7 by Sasha.

graded

② Equivariant homology

A cpx torus

$\mathfrak{a} = \mathrm{Lie}\ A$

$\chi^*(A) \otimes_{\mathbb{Z}} \mathbb{C} \xrightarrow{\sim} \mathfrak{a}^*$

$$S(\mathfrak{a}) = \mathrm{Sym}(\mathfrak{a}) = \bigoplus_{d=0}^{\infty} \mathrm{Sym}^d(\mathfrak{a}^*)^*$$

cpx valued pol-functions on $\mathfrak{a}^*$.

$\varphi : A \to \mathbb{C}^\times$ character. $\to$ line bundle on BA $\mathcal{L}_\varphi$.
w/ 1st Chern class $c_1(\varphi)$

$$\mathcal{S}(\mathfrak{a}) \xrightarrow{\sim} H^*(BA) = H^*_A(pt)$$ [Chern-Weil iso]

$A \times A \to A$ multiplication induces $m : BA \times BA \to BA$.
endows $H_*(BA)$ w/ algebra structure.

$\quad\ \ \Vert$
$H^A_*(pt)$

cap product $\quad H^A_*(pt) \otimes H^*_A(pt) \to H^A_*(pt)$

module over

Hopf formula $\quad x \in H^2(BA)$
$\qquad\qquad\qquad\quad \Vert$
$\qquad\qquad\qquad c_1(\mathcal{L})$

$m^*\mathcal{L}$ is a line bundle over $A \times A$ associated to the character $\varphi \circ m$ whose derivative at 1 is

$$A\times A \to \mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m \qquad d\varphi_{(1,1)}$$

$$\begin{aligned}\varphi\big((1+h)(1+h')\big) &= \varphi(1+h+h') \\ &= 1+d\varphi_1(h+h')\end{aligned}$$

So $m^* x = x\otimes 1 + 1\otimes x$

so x acts on $H_*^A(pt)$ as a derivation:

$$\begin{aligned}(x \cap m(y,z)) &= x \cap m_*(y\otimes z) \\ &= m_*\big(m^*(x) \cap (y\otimes z)\big) \\ &= m_*\big((x\otimes 1)\cap(y\otimes z) + (1\otimes x)\cap(y\otimes z)\big) \\ &= m_*\big((x\cup y)\otimes z\big) + m_*\big(y\otimes(x\cup z)\big) \\ &= m(x\cup y, z) + m(y, x\cup z)\end{aligned}$$

$\left(\begin{aligned} f_*(f^* c \cap \sigma) \\ = c \cap f_*\sigma\end{aligned}\right)$ (naturality of cap-product)

$$S(\mathfrak{a}) \xrightarrow{\text{inverse}} H_*^A(pt) \text{ dual to Chern Weil}$$

$$\begin{array}{ccc} \partial_k \otimes S_j & \longrightarrow & S_{j-k} \\ \wr\!\!| & \circlearrowleft & \wr\!\!| \\ H^{2k}_A(pt) \otimes H_{2j}^A(pt) & \xrightarrow{\cap} & H^A_{2j-2k}(pt) \end{array}$$

Change of torus

$T \subset A$ $\qquad BT \to BA$

$$I = \ker\left(H^*_A(pt) \to H^*_T(pt)\right)$$

$$H^*_T(X) = H^*_A(X) \otimes_{H^*_A(pt)} H^*_T(pt) = \frac{H^*_A(X)}{I H^*_A(X)}$$

[purity of $X \Rightarrow X$ equivariantly formal for any torus acting]

$$H^*_V(X) = H^*(X) \otimes H^*_V(pt).$$

$H^A_*(pt)$ is a module over $\mathcal{D} = H^*_A(pt)$

and $H^T_*(X) = H^A_*(X)\{I\}$ elements of $H^A_*(X)$ annihilated by the homogeneous ideal I

by $H^*_A(pt)$-duality.

- A character thing: $\varphi : A \to \mathbb{C}^\times$

$\partial\varphi \in \mathfrak{a}^* \subset \mathcal{D}(\mathfrak{a})$

$\mathfrak{m} = \ker \partial\varphi \subset \mathfrak{a}$

$j : pt/_{\mathfrak{n}} \to pt/_A$ induces

pull-back $\mathcal{D}(\mathfrak{a}) \longrightarrow \mathcal{D}(\mathfrak{m})$ w/ kernel ∂_φ.

Dualizing, we get $j^* : S(\mathfrak{m}) \to S(\mathfrak{a})$ w/ image polynomials annihilated by $\partial\varphi$.

extend the A-action to $\mathbb{P}^1$.

$$0 \to H^A_{*+1}(\mathbb{CP}^1, \{0\} \cup \{\infty\}) \xrightarrow{\partial} H^A_*(\{0\}) \oplus H^A_*(\{\infty\}) \to H^A_*(\mathbb{CP}^1) \to 0.$$

$$\begin{array}{ccc} \| & & \| \\ S(\mathfrak{m}) & \xrightarrow{\delta} & S(\mathfrak{a}) \oplus S(\mathfrak{a}) \\ f & \longmapsto & (j^*(f), -j^*(f)) . \end{array}$$

use that $\ker \varphi$ is a subtorus of A (and then that $H^*_T(\mathbb{C}^*) \simeq S(\mathfrak{m}^*)$.

[instead it's not true ? or the only possible problem comes from torsion ?]

$$H^A_*(\mathbb{CP}^1, \{0\} \cup \{\infty\}) = H^A_*(\mathbb{C}^*) \simeq S(\mathfrak{m}^*)$$

At the level of singular chains

dist. triangle

$$\begin{tikzcd} \text{pair} & C^A_*(\{0\} \cup \{\infty\}) \arrow[r] \arrow[d, "\text{qiso}"] & C^A_*(\mathbb{P}^1) \arrow[r] \arrow[d, equal] & C^A_*(\mathbb{P}^1, \{0,\infty\}) \\ \text{Mayer-Vietoris} & C^A_*((\mathbb{P}^1 \setminus \{0\}) \sqcup (\mathbb{P}^1 \setminus \infty)) \arrow[r] & C^A_*(\mathbb{P}^1) \arrow[r] & C^A_*(\mathbb{P}^1 \setminus \{0,\infty\})[1] \arrow[u, "\text{qiso}"'] \\ & C^A_*(\mathbb{P}^1 \setminus \{0,\infty\}) \arrow[u] & & \end{tikzcd}$$

pushforward for Lmaps is dual to pullback in cohomology

explaining the formula

+ connection morphism

Lemma (Chang – Skjelbred, homological version)

$A \curvearrowright Y$ cpx (pure) proj variety

$Y_0 \subset Y$ fixed points

$Y_1 \subset Y$ union of 0 & 1 dim orbits (= 1-dim skeleton)

$$H^A_*(Y_1, Y_0) \xrightarrow{\partial} H^A_*(Y_0) \longrightarrow H^A_*(Y) \longrightarrow 0 \quad \text{is exact.}$$

If $Y_0 = \{y_1, \dots, y_r\}$ finite

$Y_1 = \{E_1, \dots, E_d\}$ finite too, $\quad \partial E_i = \{y_{i_a}\} \cup \{y_{i_b}\}$.

$m_i = \mathrm{Lie}(\mathrm{Stab}\, E_i)$

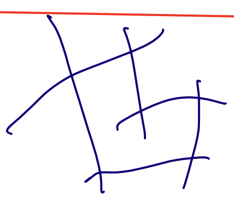
$j_i^* : S(m_i) \longrightarrow S(\mathfrak{a})$.

complex obtained from the 1-D skeleton.

$$\bigoplus_{i=1}^{d} S(m_i) \xrightarrow{\beta} \bigoplus_{k=1}^{r} S(\mathfrak{a}) \longrightarrow H^A_*(Y) \longrightarrow 0$$

$$\beta_i(f_i) = (0 \dots 0\ j_i^* f_i, \longrightarrow -j_i^*(f_i)\ 0 \dots 0)$$

$$\beta = \sum \beta_i$$



bunch of $\mathbb{P}_1$'s. Reproduce the previous reasoning to this situation.

Thm (GKM - localization of equivariant cohomology
SERG Reading group Talk 2
+ homology version)

Cohomology version

Under mild assumptions, $H_T^*(X)$ is determined by fixed points and 1-dimensional orbits.

X cpx proj var

$\circlearrowleft$

$T \simeq (\mathbb{C}^*)^n$ finite number of fixed points, finite number of 1-dim orbits

E_j w/ $\overline{E_j} = E_j \cup \{x_{j_0}, x_{j_\infty}\}$

X equivariantly formal

$$H_T^*(X) \xrightarrow{\iota^*} H_T^*(X^T) \simeq \bigoplus_{x \in X^T} S(t^*) \quad \text{is injective}$$

w/ image

$$H = \left\{ (f_1, \dots, f_k) \in H_T^*(X^T) \;\middle|\; f_{j_0}|_{\ell_j} = f_{j_\infty}|_{\ell_j} \ \forall 1 \leq j \leq e \right\}$$

$\ell_j = \operatorname{Ker}\left[\square_j : T \to \mathbb{C}^*\right)$

character of the T-action on $\overline{E_j}$

superfluous.

Affine Springer fibers

$X_*(T)$

dual torus $\hat{T} = \mathrm{Hom}(X_*(T), \mathbb{G}_m)$

$T \subset G$, $\hat{G}$. $(X_*(T), \phi^\vee, X^*(T), \phi)$.

$(X^*(T), \phi, X_*(T), \phi^\vee)$

$F = \mathbb{C}((\varepsilon))$

$\mathcal{O} = \mathbb{C}[[\varepsilon]]$ ring of integers.

Valuation

$T \cong (\mathbb{C}^*)^n$ $\mathbb{C}$-torus.

$$1 \to T(\mathcal{O}) \to T(F) \xrightarrow{\mathrm{val}} X_*(T) \to 1$$

is exact,

$$\alpha(\mathrm{val}(\ell)) = \mathrm{val}(\alpha(\ell)), \quad \alpha(\ell) \in F^\times$$

$\alpha \in X^*(T)$

uniformizing parameter ε gives splitting

$$\begin{array}{ccccc} \mathbb{Z}^n & \simeq & X_*(T) & \longrightarrow & T(F) \\ (\beta_1, \dots, \beta_n) & = & \beta & \longmapsto & \beta(\varepsilon) \\ & & & & \| \\ & & & & (\varepsilon^{\beta_1}, \dots, \varepsilon^{\beta_n}) \end{array}$$

image = "lattice of translations".

free abelian group of rk dim T

$\mathbb{G}_m \to T$

$z \mapsto (z^{\beta_1}, \dots, z^{\beta_n})$

Affine Springer fiber

G conn red $/\mathbb{C}$ $\qquad$ Lie $G = \mathfrak{g}$

$\mathfrak{g}(F) = \mathfrak{g} \otimes_{\mathbb{C}} F$

$\mathfrak{g}(\mathcal{O})$

$K = G(\mathcal{O})$

$X = G(F)/G(\mathcal{O})$ affine Grassmannian

ind algebraic variety.

$H \subset G$ conn red alg subgroup

$$H(K)/H(\mathcal{O}) \hookrightarrow X.$$

$$\left(\overset{\text{Key fact}}{G(\mathcal{O}) \cap H(K) = H(\mathcal{O})}. \right)$$

$\gamma \in \mathfrak{g}(F)$ gives rise to the affine Springer fiber

$$X_\gamma = \{ xK \in G(F)/K \mid \mathrm{Ad}(x^{-1})(\gamma) \in \mathfrak{g}(\mathcal{O}) \}.$$

γ "compact" if $X_\gamma \neq \emptyset$.

KL88: $\gamma \in \mathfrak{g}(F)$ is rss iff X_γ is finite dimensional ind-subvariety of X.

Purity conjecture: If $\gamma \in \mathfrak{g}(F)$ is compact, regular, semisimple then $(\forall i)$, $H^i(X_\gamma; \mathbb{C})$ is pure of weight i.

Purity of equivalued affine Springer fibers GKM 2003

$k = \bar{k}$, G conn red/k
$\cup$
A max torus

$\mathfrak{a} = X_*(A) \otimes_{\mathbb{Z}} \mathbb{R}$

$F = k((\varepsilon)) \supset \mathcal{O}$.

$\bar{F}$ $\underline{G} = G(F)$

$y \in \mathfrak{a}$ a single apartment $\underline{\mathfrak{g}} = \mathfrak{g}(F)$

$\mathcal{G}_y, \mathfrak{g}_y$ connected parahoric subgroup / algebra.

$\bar{\Phi} \subset X^*(A)$ roots of G

$\Phi = \{\bar{\alpha} + n \mid \bar{\alpha} \in \bar{\Phi}, n \in \mathbb{Z}\}$ affine roots

$\bar{\alpha} \in \bar{\Phi}$ gives $L_{\bar{\alpha}} : \mathbb{G}_a \xrightarrow{\sim} \mathfrak{u}_{\bar{\alpha}} \subset \mathrm{Lie}\, G$.

$\alpha = \bar{\alpha} + n$ affine root $\rightsquigarrow \mathfrak{g}_\alpha = L_{\bar{\alpha}}(t^n k) \subset \underline{\mathfrak{g}}$.

$$\mathfrak{g}_{\alpha + \mathbb{Z}_{\geq 0}} := L_{\bar{\alpha}}(t^n \mathcal{O})$$

$y \in \mathfrak{a} \rightsquigarrow$ $\mathfrak{g}_y$ = sub lie alg gen by $\{\mathfrak{g}_{\alpha + \mathbb{Z}_{\geq 0}}, \alpha \in \Phi, \alpha(y) \geq 0\}$.

Similarly for $\mathcal{G}_y$.

$\mathcal{F}_y := \underline{G} / \mathcal{G}_y$. $y = 0$: affine Grassmannian

$u \in \mathfrak{g}$ determines a closed subset

$$\boxed{\mathcal{F}_y(u) := \{ g \in G/G_y \mid \mathrm{Ad}(g^{-1})(u) \in \mathfrak{g}_{y,r} \}}$$

"affine Springer fiber"

T max F-torus in G, $\mathfrak{t}$

$\cup$

A_T max split

$u \in \mathfrak{t}$ "integral" if $\mathrm{val}(d\lambda(u)) \geq 0 \quad \forall \lambda \in X^*(T)$

regular

$T(F) \overset{?}{=} Z_G(u) \circlearrowright \mathcal{F}_y(u)$

$\cup \qquad \varepsilon^{\mu}$

$\Lambda := X_*(A_T) \ni \mu$

Λ acts freely on $\mathcal{F}_y(u)$

KL: $\Lambda \backslash \mathcal{F}_y(u)$ is projective $/k$, $\neq \emptyset \iff u$ integral, + formula dim. conjectured (proved by Bezr...)

normalization: $\mathrm{val}(\varepsilon) = 1$

$u \in \mathfrak{t}^{\mathrm{reg}}$ is equivalued with valuation $s \in \mathbb{Q}$ if

$\mathrm{val}\, \alpha(u) = s \quad \forall \alpha$ root of T over $\bar{F}$ and

redundant for adjoint groups [$\mathrm{val}(d\lambda(u)) \geq s \quad \forall \lambda \in X^*(T)$.

Thm: Assume $\# W \in k^{\times}$.

absolute Weyl group of G.

u regular equivalued of $\mathfrak{t}$. Then $\mathcal{F}_y(u)$ admits a Hessenberg paving

Hessenberg paving: X ind-scheme

$X_0 \subset X_1 \subset \cdots$ exhausting filtration

$X_i \setminus X_{i-1}$ disjoint union of iterated aff space bundles over Hessenberg varieties.

$G(F)$-conj. classes of max F-tori in G are par. by conj classes in W.

T Coxeter if the corresponding conjugacy class consists of Coxeter elements = product of all simple reflections.

$M = Z_G(A_T)$ is Levi subgrp of G, split $/_F$.

$\overset{\cup}{T}$ max F-torus

T "weakly Coxeter" if Coxeter in M.

split max tori are weakly Coxeter; all max tori in GL_n are weakly Coxeter.

Hessenberg paving obtained by intersecting aff Springer fiber w/ orbits of parahoric subgrp depending on T.

If parahoric is Iwahori (e.g. T weakly Coxeter), resembles

- CLP88 for ordinary Springer fibers
- coincide w/ LS91, homogeneous & EL, T Coxeter torus in GL_n.

Thm: $\# W \in k^\times$, T weakly Coxeter.

$u \in \mathfrak{t}$ reg integral equivalued. ⟶ weaker than "homogeneity" (guarantees that loop rotation preserves the Springer fiber.

$\mathcal{F}_y(u)$ admits a paving by affine spaces.

Not GL_n: many pavings by non-affines

app BK of KL: one irr comp of SF of a hom elt of $sp(6)$ has dom map to ell curve.

More general (affine) Springer fibers

$u \in \mathfrak{g}$ replaced by $v \in \underline{V} := V \otimes_k F$, V f dim rep of G over k.

$\mathfrak{g}_y$ played by lattices $\underline{V}_{y,t}$ in $\underline{V}$, $t \in \mathbb{R}$, analogous to Moy-Prasad lattices in $\mathfrak{g}$.

$\to$ generalized ASF $\mathcal{F}_y(t, v)$.

applications in GGP unramified case

Laumon lemme fondamental pr les groupes unitaires.

TBC

Bruhat decomposition : $T \subset G$ max torus $/\mathbb{C}$ ($\Rightarrow$ split over F)

$$\Lambda \subset T(F) \text{ translation lattice}$$
$$\overset{\wr}{T(F)/T(\mathcal{O})}$$

$$\phi : \Lambda \longrightarrow G(F)/G(\mathcal{O})$$

$B \supset T$ Borel; $I \subset G(F)$ Iwahori

$G(F) = I \Lambda K$

$$\begin{tikzcd} I \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & B \arrow[d] \\ G(\mathcal{O}) \arrow[r, "\text{evaluation at } \varepsilon = 0."'] & G \end{tikzcd}$$

for $G = GL_n$, reduction of matrices of $\mathbb{C}[[\varepsilon]]$ principal ring.

$$\Rightarrow \boxed{\underset{\text{aff Grass}}{X} = \bigsqcup_{l \in \Lambda} IlK/K}.$$

$x_0 = K/K \subset X$

$C_l = Ilx_0 \subset X$ cell.

loop rotation: $\mathbb{G}_m \curvearrowright F$ preserving $\mathcal{O}$.

$$\mathcal{O}^{\mathbb{G}_m} = \mathbb{C}\cdot\varepsilon^0.$$
$$\Vert$$
$$F^{\mathbb{G}_m}.$$

$T(\mathbb{C}) \subset I.$

$\mathbb{G}_m \curvearrowright X$ commutes w/ $\overset{\Vert}{T}$-action.

Extended torus: $\tilde{T}(\mathbb{C}) = T(\mathbb{C}) \times \mathbb{C}^\times$ preserves the
loop rotations
Bruhat decomposition. translation lattice

$$(t, \lambda) \cdot g\, \ell K = \lambda(tgt^{-1})\, \ell K$$

since $T(\mathbb{C}) \curvearrowright G(F)/G(\mathcal{O})$ by conjugation,

Affine roots: $\Phi = \Phi(G,T) \supset \Phi^+$ B Borel.

$$\mathfrak{g}(\mathbb{C}) = \mathfrak{t}(\mathbb{C}) \oplus \bigoplus_{\alpha \in \Phi} \mathbb{C} Y_\alpha$$ root space dec.

$$\tilde{\Phi} = \{(\alpha, k) : \alpha \in \Phi, k \in \mathbb{Z}\}$$ affine roots.

involves character of $\tilde{T}(\mathbb{C})$ by $(\alpha, k)(t, \lambda) = \alpha(t)\lambda^k$.

$\tilde{T}(\mathbb{C})$ acts on the affine root space $\mathbb{C}\varepsilon^k Y_\alpha$ through this character.

C_0 fundamental alcove determined by I

$\tilde{\Delta}$ simple affine roots $= \{\alpha_1, \dots, \alpha_n\} \cup \{\alpha_0\}$
$\alpha_0 = -\theta + \delta$

θ = longest positive root of Δ

$$C_0 = \{a \in \underset{\underset{\mathrm{Lie}\, T_{\mathbb{R}}}{\cong}}{X_*(T)} \otimes_{\mathbb{Z}} \mathbb{R} : \alpha(a) + k > 0 \ \ \forall (\alpha, k) \in \tilde{\Delta}\}$$

Split s.e.s.: $1 \to I_+ \to I \to T(\mathbb{C}) \to 1$ (picture for SL_2)

I_+: unipotent radical of I; $T(\mathbb{C}) \to I$: diagonal embedding.

$$\mathfrak{n}(I) = \mathrm{Lie}(I_+) = \prod_{n \in \mathbb{Z}_{>0}} \varepsilon^n \mathfrak{t}(\mathbb{C}) \oplus \prod_{\substack{(\alpha,k) \in \tilde{\Phi} \\ \alpha(a)+k>0 \\ \forall a \in C_0}} \mathbb{C}\varepsilon^k Y_\alpha$$

[mistake in 5.6.2 of GKM]

Lemma (Bruhat cells)

$x_0 = K/K \in X$

$\ell \in \Lambda$

exp map: $C_\ell = I\ell x_0 \simeq D_\ell := \bigoplus_{\substack{(\alpha,k) \in \tilde{\Phi} \,|\, \mathrm{val}(\alpha(\ell))+k<0 \\ \alpha(a)+k>0 \; \forall a \in C_0}} \mathbb{C}\varepsilon^k Y_\alpha$ (finite set) $(*)$

$T(\mathbb{C})$-equiv obviously

exp commutes w/ loop rotation.

$\tilde{T}(\mathbb{C})$-equivariant isomorphism.

Proof: analogous to the classical Bruhat decomposition.

I_+ acts transitively on C_ℓ (since $T(\mathbb{C}) \subset I$ commutes with Λ) (ck)

ℓx_0 is stabilized by $I_+ \cap \ell G(\mathcal{O}) \ell^{-1}$ whose Lie algebra is connected

$(g\ell K = \ell K \Leftrightarrow \ell^{-1} g \ell \in K)$

sum of affine root spaces $\mathbb{C}\varepsilon^k Y_\alpha$

s.t. $\alpha(a)+k>0 \quad a \in A$

& $\boxed{Ad(\ell^{-1})(\varepsilon^k Y_\alpha) \in \mathfrak{g}(\mathcal{O})}$

$\Updownarrow \quad k - val(\alpha(\ell)) \geq 0$

$$\left(\left(Ad(\ell^{-1})Y_\alpha = \varepsilon^{-val(\alpha(\ell))} Y_\alpha\right)\right)$$

$Lie(\ell G(\mathcal{O}) \ell^{-1})$

$= \bigoplus \mathbb{C}\varepsilon^k Y_\alpha$

$Ad(\ell^{-1})(\varepsilon^k Y_\alpha) \in \mathfrak{g}(\mathcal{O})$

$\mathfrak{k}$ is $\tilde{T}(\mathbb{C})$-invariant complement $\Rightarrow$ exponential map takes it isomorphically to $I\ell x_0$.

I^+ is pro-unipotent group

$$\begin{array}{ccc} Lie\, I^+ & \xrightarrow{\sim} & I^+ \\ \cup & & \cup \\ Lie\, Stab & \xrightarrow{\sim} & Stab. \end{array}$$

is isomorphism w/ inverse given by logarithm.

Fixed points

$X^{\mathbb{C}^\times} = G(\mathbb{C}) \Lambda G(\mathcal{O})/G(\mathcal{O})$

fixed points of loop rotation

$X^{T(\mathbb{C})} = \Lambda x_0$.

torus fixed points

<u>Proof</u>: ① • $G(\mathbb{C})\Lambda G(\mathcal{O})/G(\mathcal{O}) \subset X^{\mathbb{C}^\times}$ clearly.

• loop rotation preserves Bruhat cells.

• suffices to show $X^{\mathbb{C}^\times} \cap \underset{\underset{I^+ \ell G(\mathcal{O})}{\shortparallel}}{C_\ell} \subset G(\mathbb{C})\Lambda x_0$

$\forall \ell \in \Lambda$.

Fixed points of $\mathbb{C}^\times$ on D_ℓ need $k=0$, so exponential is in $G(\mathbb{C}) \cap I$ ✓

② • $\Lambda x_0 \subset X^{T(\mathbb{C})}$ clearly

• $C_\ell^{T(\mathbb{C})} \simeq D_\ell^{T(\mathbb{C})} \simeq \{0\}$

so $T(\mathbb{C})$ acts on C_ℓ with only fixed point ℓx_0. ∎

One dimensional orbits $X_1 \subset X$ 1-dim skeleton of $T(\mathbb{C})$.

$\alpha \in \Phi^+ \rightsquigarrow U_\alpha$ 1-dim unip subgrp.

$\alpha \in \Phi^+ \rightsquigarrow$ red. conn. alg subgrp $H_\alpha \subset G$ of semisimple rk 1

$H_\alpha = \langle T, U_\alpha, U_{-\alpha} \rangle$.

$X^\alpha = H_\alpha(F)/H_\alpha(\mathcal{O}) \hookrightarrow X$.

Lemma (1-dimensional orbits of $T(\mathbb{C})$)

$$X_1 = \bigcup_{\alpha \in \Phi^+} X^\alpha$$

If $\alpha \neq \beta \in \Phi^+$, $\boxed{X^\alpha \cap X^\beta = \Lambda}$

Proof: • $\dim T = 1 \Rightarrow$ a single root (or none) $\leadsto$ both sides coincide w/ X.

• Assume $\dim T \geq 2$

$T(\mathbb{C}) \curvearrowright H_\alpha$ factors through 1-dim quotient $T(\mathbb{C})/\ker(\alpha) \Rightarrow X^\alpha \subset X_1$.

$$X_1 \subset \bigcup_{\alpha \in \Phi^+} X^\alpha \Leftarrow \left[X_1 \cap C_\ell \subset \bigcup_{\alpha \in \Phi^+} X^\alpha \cap C_\ell \right] \quad \forall \ell.$$

$\to$ determine 1-dim orbits on D_ℓ (of $T(\mathbb{C})$) $\to$ coordinate axes $\mathbb{C}\varepsilon^k Y_\alpha$.

$\alpha \in \Phi^+$ fixed.

$$D_{\ell,\alpha} = \bigoplus_{\substack{k \in \mathbb{Z} \\ \alpha(a)+k>0 \ \forall a \in C_0 \\ \mathrm{val}(\alpha(\ell))+k<0}} \mathbb{C}\varepsilon^k Y_\alpha = \varepsilon^k \mathrm{Lie}(U_\alpha)(\mathbb{C}).$$

$C_{\ell,\alpha}$ corresponding subset of the Bruhat cell.

$C_{\ell,\alpha} = X^{\alpha} \cap C_{\ell}$ = Bruhat cell of X^{α}

$$\left[\; I^{+} \in G(\mathcal{O})/G(\mathcal{O}) \supset I_{\alpha}^{+} \in H_{\alpha}(\mathcal{O})/H(\mathcal{O}) \;\right]$$

If $\alpha \neq \beta$, $D_{\ell,\alpha} \cap D_{\ell,\beta} = \{0\}$ so $X^{\alpha} \cap X^{\beta} = \Lambda$

$\to \quad X^{\alpha} = \bigsqcup X^{\alpha} \cap C_{\ell}$

$X^{\beta} = \bigsqcup X^{\beta} \cap C_{\ell}$ $\blacksquare$

SL(2) ASF

$G(\mathbb{C}) = SL(2,\mathbb{C})$ $\qquad X^{SL(2)} = G(F)/G(\mathcal{O}) \ni x_0 = K$.

$T = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$, $\quad X^*(T) \ni \alpha \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = a^2$ pos. root.

$\alpha^{\vee} : F^{\times} \to T(F)$ corresponding coroot:

$$\alpha^{\vee}(b) = \begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix}.$$

$$\Lambda^{SL(2)} = \langle \alpha^\vee(\varepsilon) \rangle \qquad \ell_m = \alpha^\vee(\varepsilon^m)$$

$$\alpha^\vee(\varepsilon) = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix}$$

$n \leq -1$ $\qquad x_n = \begin{pmatrix} 1 & \varepsilon^n \\ 0 & 1 \end{pmatrix} K \in X^{SL(2)}.$

$\alpha' : \mathfrak{t}(F) \to F$ differential of $\alpha : T(F) \to F$.

Lemma (Nadler)

$$X^{SL(2)} = \bigsqcup_{n \leq 0} \underbrace{T(F)\, x_n}_{\text{cpx dimension } |n|}$$

$\gamma \in \mathfrak{t}(G)$, $v = \mathrm{val}(\alpha'(\gamma))$,

$$X_\gamma^{SL(2)} = \bigcup_{n=-v}^{0} T(F) \cdot x_n \qquad \text{union of } T(F)\text{-orbits.}$$

It is preserved by loop rotations.

We let $X_\gamma^{SL(2)} := X_{\leq v}^{SL(2)}$.

Proof: $x \in X^{SL(2)}$ is gK for $g = \begin{pmatrix} \varepsilon^m & b_0\varepsilon^n \\ 0 & \varepsilon^{-m} \end{pmatrix}$

where either : (1) $b_0 = 0$
(2) $b_0 \in \mathcal{O}^\times$ and $n-m<0$.

$x_0 = K = G(\mathcal{O})$

$B(F) \curvearrowright X = G(F)/G(\mathcal{O})$
transitively

$x = g x_0$, $g = \begin{pmatrix} a & b' \\ 0 & a^{-1} \end{pmatrix}$ $a, b' \in F$

$a = a_0 \varepsilon^m$ $a_0 \in \mathcal{O}^\times$

right mult. by $\alpha^\vee(a_0^{-1}) \in K$ get $g = \begin{pmatrix} \varepsilon^m & b \\ 0 & \varepsilon^{-m} \end{pmatrix}$

- $b = 0$ or $\mathrm{val}(b) < m$ ✓
- otherwise, set $b = b_0 \varepsilon^n$ $n \geq m$, $b_0 \in \mathcal{O}^\times$

right mult g by $\begin{pmatrix} 1 & -b_0\varepsilon^{n-m} \\ 0 & 1 \end{pmatrix}$.

- let $x = gK$, $g = \begin{pmatrix} \varepsilon^m & b_0\varepsilon^n \\ 0 & \varepsilon^{-m} \end{pmatrix}$.

* If $b_0 = 0$, $x = \alpha^\vee(\varepsilon^m)\, x_0 \in T(F) \cdot x_0$.
* If $b_0 \in \mathcal{O}^\times$, $m > n$, let $a \in \mathcal{O}^\times$ be a square-root of b_0.

$t = \alpha^\vee(a\varepsilon^m) \in T(F)$

$k = \alpha^\vee(a^{-1}) \in K.$

$$t\, x_{n-m}\, k = \begin{pmatrix} a\varepsilon^m & 0 \\ 0 & a^{-1}\varepsilon^{-m} \end{pmatrix}\begin{pmatrix} 1 & \varepsilon^{n-m} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix}$$

$$= \begin{pmatrix} a\varepsilon^m & a\varepsilon^n \\ 0 & a^{-1}\varepsilon^{-m} \end{pmatrix}\begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix}$$

$$= g$$

- dimension statement

stabilizer of $x_n = \begin{pmatrix} 1 & \varepsilon^n \\ 0 & 1 \end{pmatrix} K$

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} x_n = \begin{pmatrix} a & a\varepsilon^n \\ 0 & a^{-1} \end{pmatrix} K \overset{?}{=} \begin{pmatrix} 1 & \varepsilon^n \\ 0 & 1 \end{pmatrix} K$$

$$\Leftrightarrow \begin{pmatrix} 1 & -\varepsilon^n \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & a\varepsilon^n \\ 0 & a^{-1} \end{pmatrix} \in K$$

$$\|$$

$$\begin{pmatrix} a & \varepsilon^n(a-a^{-1}) \\ 0 & a^{-1} \end{pmatrix} \in K$$

- $a \in \mathcal{O}^\times$

$n \leq 0$

- $\mathrm{val}(a-a^{-1}) \geq -n$

$\gamma \in \mathfrak{t}(\mathcal{O})$ $\quad X_\gamma^{SL(2)} = \{ gK \mid g^{-1}\gamma g \in \mathfrak{g}(\mathcal{O}) \}$

is preserved by $T(F)$.

$$\gamma = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \in \mathfrak{t}(\mathcal{O}).$$

$$\mathrm{Ad}(x_n)\gamma = \begin{pmatrix} a & -2a\varepsilon^n \\ 0 & -a \end{pmatrix} \in \mathfrak{g}(\mathcal{O}) \iff \mathrm{val}(a) + n \geq 0$$

turning torus:

$$\begin{pmatrix} 1 & a\varepsilon^n \\ 0 & 1 \end{pmatrix}$$

$a \in \mathbb{C}^\times$

$a = b^2$

$$= \begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix} \begin{pmatrix} 1 & \varepsilon^n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} b^{-1} & 0 \\ 0 & b \end{pmatrix}.$$

- dimension $T(F)\cdot x_n$ $\qquad \begin{pmatrix} 1 & \varepsilon^n \\ 0 & 1 \end{pmatrix}$

$$\begin{pmatrix} a & a\varepsilon^n \\ 0 & a^{-1} \end{pmatrix} K = \begin{pmatrix} 1 & \varepsilon^n \\ 0 & 1 \end{pmatrix} K$$

$$\begin{pmatrix} a & \varepsilon^n(a-a^{-1}) \\ 0 & a^{-1} \end{pmatrix} \in G(\mathcal{O})$$

$a \in \mathcal{O}^\times$ $\qquad T(F)$

& $\mathrm{val}(a-a^{-1}) \geq -n$

$$t(F) \longrightarrow T_{x_n} X^{SL(2)}$$

$$\begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \longmapsto \begin{pmatrix} a & a\varepsilon^n \\ 0 & -a \end{pmatrix}$$

$G \subset X^{SL(2)}$ 1-dim orbit, contains two fixed points $l_s, l_t \in \Lambda = T(F)/T(O)$ in its closure.

Lemma: $\forall l_s, l_t \in \Lambda$, $\exists!$ 1-dim orbit G_{st} of $\tilde{T}(\mathbb{C})$ in $X^{SL(2)}$ which connects them.

The stabilizer of any point in G_{st} is the kernel of the affine root $(\alpha, s+t)$.

$$G_{st} \subset X_\gamma^{SL(2)} \Leftrightarrow val(\alpha'(\gamma)) \geq |s-t|.$$

Proof: $\mathcal{T}$ $T(O)$-orbit

$$\exists! s \in \mathbb{Z},\ \exists! n \leq 0,\quad l_s x_n \in \mathcal{T}$$

$$\mathcal{T} \subset T(F) x_n$$

$$T(F)/T(O) \simeq \mathbb{Z}.$$

$$T(F) = \bigsqcup_{s \in \mathbb{Z}} T(O) l_s.$$

We prove: $t = s + n$

(1) the $\tilde{T}(\mathbb{C})$-orbit G of $l_s x_n$ is 1-dim

(2) this is the only 1-dim orbit of $\tilde{T}(\mathbb{C})$ in $\mathcal{T}$

(3) G connects l_s and $l_t = l_s + n\alpha^\vee(\varepsilon)$

(4) $\tilde{T}(\mathbb{C})$ acts on G through the affine root $(\alpha, 2s+n)$.

$\tilde{T}(\mathbb{C})$-action on the $T(\mathcal{O})$-orbit of $l x_n$.

$$\alpha^\vee(b) \in T(\mathcal{O})$$

$$b = \sum_{i \geq 0} b_i \varepsilon^i$$

$a \in \mathbb{C}^\times$.

$$l_s x_n = \begin{pmatrix} \varepsilon^s & \varepsilon^{s+n} \\ 0 & \varepsilon^{-s} \end{pmatrix}$$

$(\alpha^\vee(a), \lambda) \in \tilde{T}(\mathbb{C}) \curvearrowright \alpha^\vee(b) l_s x_n$ is the point

$y =$ TBFinished.

Equivariant homology of $SL(2)$-Springer fibers

$\mathcal{D} = \mathcal{D}(\mathfrak{t})$ = diff ops on $\mathfrak{t}^*$

$\simeq H^*_T(pt)$ via Chern-Weil.

$S(\mathfrak{t})$ symm algebra of $\mathbb{C}$-valued $\overset{pol}{\text{fcts}}$ on $\mathfrak{t}^*$

character $\alpha \rightsquigarrow \partial_\alpha \in \mathcal{D}(\mathfrak{t})$ diff op

$S(\mathfrak{t})\{\partial_\alpha^d\}$ kernel of ∂_α^d

Chern-Weil : $H^T_*(\Lambda) \cong \mathbb{C}[\Lambda] \otimes_{\mathbb{C}} S(\mathfrak{t})$. of $\mathcal{D}$-modules.

$\gamma \in \mathfrak{t}(\mathcal{O})$

Proposition : $v = \mathrm{val}(\alpha'(\gamma))$. The inclusion $\Lambda \subset X$ induces exact sequences:

$$\begin{array}{ccccccc}
H^{\tilde{T}}_*(X_\gamma, \Lambda) & \xrightarrow{\tilde{\beta}} & H^{\tilde{T}}_*(\Lambda) & \longrightarrow & H^{\tilde{T}}_*(X_\gamma) & \longrightarrow & 0 \\
\cup & & \cup & & \cup & & \\
H^T_*(X_\gamma, \Lambda) & \xrightarrow[\beta]{} & H^T_*(\Lambda) & \longrightarrow & H^T_*(X_\gamma) & \longrightarrow & 0
\end{array}$$

and the image of β is the $\mathcal{D}$-submodule

$$\sum_{d=1}^{v} (1-\alpha^\vee)^d \, \mathbb{C}[\Lambda] \otimes S(t) \{ \partial_\alpha^d \} \subset \mathbb{C}[\Lambda] \otimes S(t).$$

The coroot $\alpha^\vee$ determines canonical isomorphisms:

$$S(\tilde{t}) \cong \mathbb{C}[x, t]$$

$$\mathcal{D}(\tilde{t}^\vee) \cong \mathcal{D}(\partial_x, \partial_t)$$

$\forall\ a, b \in \mathbb{Z}$ s.t. $|b-a| \leq v = \mathrm{val}(\alpha'(\gamma))$, $\exists!$ 1-dim orbit $E_{a,b} \subset X_\gamma$ connecting l_a and l_b and $\tilde{T}$ acts through the character ϕ_{ab} whose differential

$$\phi_{a,b} : \tilde{t} \to \mathbb{C} \quad \in t^*$$

corresponds to the differential operator

$$\begin{aligned} \partial_{ab} &= \partial_\alpha + (a+b)\, \partial_t \\ &= 2\partial_x + (a+b)\, \partial_t \end{aligned}$$

$$S(m_{ab}) = S(\ker \phi_{ab}) \hookrightarrow S(\tilde{t})$$

$j_{ab}^* S(m_{ab}) =$ pol fcts on $\tilde{t}^*$ annihilated by ∂_{ab}.

$h(x,t)$ such a function.

$$2\partial_x h(x,t) + (a+b)\partial_t h(x,t) = 0$$

$$g((a+b)x - 2t)$$

$$2(a+b)\, g'(\dots) + (a+b)(-2) g'(\dots) = 0$$

---

* § 12.6
* Proposition 12.7

Groups of semisimple rank one

Lemma: H connected reductive cpx lin alg group of rk n and ss rank 1. Then either

(1) $(\mathbb{C}^\times)^{n-1} \times SL(2,\mathbb{C})$

(2) $(\mathbb{C}^\times)^{n-1} \times PGL(2,\mathbb{C})$

(3) $(\mathbb{C}^\times)^{n-2} \times GL(2,\mathbb{C})$

Lemma:

(1) $$X^H = \bigsqcup_{\ell \in \Lambda^H/\langle \alpha^\vee \rangle} \ell\, \phi_\alpha (X^{SL(2)})$$

(2) $T^H(G)$-orbits on X^H coincide w/ $T^{SL(2)}$-orbits on X^H.

(3) $\gamma \in t^H(G)$ regular

Then $$X^H_\gamma = \bigsqcup_{\ell \in \Lambda^H/\langle \alpha^\vee \rangle} \ell\, \phi_\alpha (X^{SL(2)}_{\leq v}).$$

$v = \mathrm{val}(\alpha'(\gamma))$.

$$(1 + a_1 t + a_2 t^2 + \dots)^2$$

$$= 1 + 2a_1 t + (a_1 + 2a_2) t^2 + \dots$$

Equivariant homology of ASF

$$L_{\alpha,\gamma} = \sum_{d=1}^{\mathrm{val}(\alpha'(\gamma))} (1-\alpha^\vee)^d \, \mathbb{C}[\Lambda] \otimes S(\mathfrak{t}) \{ \partial_2^d \gamma \}.$$

<u>Thm</u>: Suppose $H_*(X_\gamma;\mathbb{C})$ is pure. Then the inclusion $\Lambda \subset X_\gamma$ induces an exact sequence

$$0 \to \sum_{\alpha \in \Phi^+} L_{\alpha,\gamma} \to \mathbb{C}[\Lambda] \otimes S(\mathfrak{t}) \to H_*^{T(\mathbb{C})}(X_\gamma) \to 0$$

Combinatorial lemmas

$d, m \in \mathbb{Z}$, $d \geq 1$, Λ free abelian group of rank one.

gen. $\alpha^\vee \in \Lambda$ determines $\ell : \Lambda \cong \mathbb{Z}$. $\ell_a \in \mathbb{Z}$. Mult by $\ell_1 = \alpha^\vee$ acts as shift operator.

$\mathbb{Q}[\partial_x, \partial_t] \curvearrowright \mathbb{Q}[\Lambda] \otimes \mathbb{Q}[x,t]$, $\ker \partial_t = \mathbb{Q}[\Lambda] \otimes \mathbb{Q}[x]$.

$$f_{m,d} = \sum_{m \leq a < b \leq m+d} c_{ab} (\ell_b - \ell_a) \otimes ((a+b)x - t)^{d-1} \in \mathbb{Q}[\Lambda] \otimes_{\mathbb{Q}} \mathbb{Q}[x,t]$$

$$c_{ab} = (-1)^{a-b} (a-b) \binom{d}{a-m} \binom{d}{b-m}.$$

$J_r \subset \mathbb{Q}[\Lambda] \otimes_{\mathbb{Q}} \mathbb{Q}[x,t]$ span:

$$J_v = \sum_{d=1}^{v} \sum_{m\in\mathbb{Z}} \mathbb{Q} f_{m,d}.$$

Lemma: $d, m \in \mathbb{Z}$, $d \geq 1$. Then

$$f_{m,d} = (-1)^d d! \, (1-\alpha^v)^d \, \ell_m \otimes x^{d-1} \in \ker(\partial_t)$$

If $v \geq 1$,
$$J_v = \sum_{d=1}^{v} (1-\alpha^v)^d \, \mathbb{Q}[\Lambda] \otimes_{\mathbb{Q}} \mathbb{Q}[x]\{\partial_x^d\}.$$

(Hall, Combinatorial Theory, 1986)

Proof: $\forall$ polynomial p,

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} p(k) = \begin{cases} 0 & \deg(p) \leq n-1 \\ (-1)^n n! & p(k) = k^n \end{cases}$$

$$\cancel{a} - \cancel{b} + \cancel{d} - \cancel{1} - \cancel{j}$$
$$= b' + a - m$$
$$= \cancel{b} + \cancel{d} - \cancel{1} - \cancel{j} - \cancel{2}m + \cancel{1}$$

Lemma 12.4: $d, h, v \geq 1$, $v \leq h$
$m \in \mathbb{Z}$

(*) $$g = \sum_{\substack{m \leq a < b \leq m+h \\ b-a \leq v}} (e_b - e_a) \otimes G_{ab}\left((a+b)x - t\right)^{d-1}$$

$G_{ab} \in \mathbb{Q}$, $b > a$.

If $\partial_t g = 0$ & $d > v$, then $g = 0$

Proof: (forme) $g_{ab} = G_{ab}\left((a+b)x - t\right)^{d-1}$

Sum (*) is $$\sum_{a=m}^{m+h-1} \sum_{b=a+1}^{\min(m+h,\, a+v)}$$

or $$\sum_{b=m+1}^{m+h} \sum_{a=\max(m,\, b-v)}^{b-1}$$

$\partial_t g = 0$

$$\sum_{b=\max(m,a-v)}^{a-1} G_{ba}(a+b)^j - \sum_{b=a+1}^{\min(m+h,\, a+v)} G_{ab}(a+b)^j = 0$$

$0 \leq j \leq d-2.$

$(a+b)^0 \quad \cdots \quad (a+b)$

$\vdots$

12.6 $P_v \subset \mathbb{Q}[\Lambda] \underset{\mathbb{Q}}{\otimes} \mathbb{Q}[x,t]$ vector space spanned by

$$(\ell_b - \ell_a) \otimes g_{ab}((a+b)x - t)$$

g_{ab} polynomials

$|b-a| \leq v$

$P_v\{\partial_t\} = P_v \cap \ker(\partial_t)$.

Proposition: Fix $v \geq 1$. Then $P_v\{\partial_t\} = J_v$:

$$\ker(\partial_t) \cap \sum_{|b-a| \leq v} \mathbb{Q}(\ell_b - \ell_a) \otimes \mathbb{Q}[(a+b)x - t]$$

$$= \sum_{d=1}^{v} (1-x^v)^d \, \mathbb{Q}[\Lambda] \otimes \mathbb{Q}[x]\{\partial_x^d\}.$$

Proof: $\supset$ is easy

- $\subset$ $P_v\{\partial_t\}$ vector subspace of P_v spanned by

$$(\ell_b - \ell_a) \otimes g_{ab}((a+b)x - t)$$

with $g_{ab} = G_{ab} z^h$ homogeneous of degree h.

$G_{ab} \in \mathbb{Q}$.

$P_{v,h}\{\partial_t\} = P_{v,h} \cap \ker(\partial_t)$.

Then $\boxed{P_v\{\partial_t\} = \sum_{h \geq 0} P_{v,h}\{\partial_t\}.}$

Lemma 12.4 $\Rightarrow$ $P_{v,h}\{\partial_t\} = 0$ $\forall h \geq v$

Need to show $P_{v,d-1}\{\partial_t\} \subset J_v$ $\forall d \leq v$.

Both are $\mathbb{Q}[\partial_x]$-modules

$P_v\{\partial_t\}$ and J_v

Homology of affine Springer fibers in the affine flag manifold

$F = \mathbb{C}((\varepsilon))$

$T \subset B \subset G$

$I \subset G(F)$ Iwahori

aff flag manifold $Y = Y^G = G(F)/I$.

$\Lambda \subset T(F)$ lattice of translation.

$\alpha^\vee \in \underset{\text{coroot}}{\phi^\vee(G, T)} \rightsquigarrow \alpha^\vee(\varepsilon) \in \Lambda$,

$\tilde{W} = \Lambda \rtimes W$.

$$Y = \bigsqcup_{w \in \tilde{W}} IwI/I \quad \text{Bruhat decomposition.}$$

Each cell has a unique fixed point of $T(\mathbb{C})$:

$$Y^{T(\mathbb{C})} \cong \tilde{W} \quad \text{comp. w/ lattice of translations.}$$

$\alpha \in \phi^+$ root $\rightsquigarrow$ $w_\alpha \in W$ reflection.

$W_\alpha = \{1, w_\alpha\}$, H^α conn. red. grp of ss rk 1 containing T and

$U_\alpha \subset G$ root subgrp.

Y^α aff flag manifold for H^α.

If $u \in Y$ $T(\mathbb{C})$ fixed point,

$$\phi_u : Y^\alpha \xrightarrow{\sim} H^\alpha \cdot u \subset Y$$

Restricts to iso of aff Springer fibers:

$$Y^\alpha_{\leq v} \cong Y_\gamma \cap H^\alpha \cdot u. \qquad \forall \gamma \in t(G),$$
$$v = \mathrm{val}\ \alpha'(\gamma).$$

$$Z^\alpha := \bigcup_{u \in W_\alpha \backslash W} H^\alpha \cdot u = \bigsqcup_{u \in W_\alpha \backslash W} \phi_u(Y^\alpha)$$

Lemma: $\gamma \in t(G)$ regular.
The union of 0 & 1-dim orbits of $T(\mathbb{C})$ in the aff Spr. fibers is

$$(Y_\gamma)_1 = \bigcup_{\alpha \in \phi^+} Z^\alpha_\gamma$$

$$Z^\alpha_\gamma = Y_\gamma \cap Z^\alpha = \bigsqcup_{u \in W_\alpha \backslash W} \phi_u(Y^\alpha_{\leq v})$$

If $\alpha \neq \beta$, $Z^\alpha_\gamma \cap Z^\beta_\gamma = \tilde{W}$.

Thm: Each $\alpha \in \Phi^+$ corresponds to a deg. one diff operator $\partial_\alpha \in \mathcal{D}(\mathfrak{t})$, $\alpha^\vee \in \Lambda$ and $w_\alpha \in W$.

$\alpha \in \Phi^+$, define the $\mathcal{D}(\mathfrak{t})$ submodule $M_{\alpha,\gamma}$ of $\mathbb{C}[\tilde{W}] \otimes_{\mathbb{C}} S(\mathfrak{t})$:

$$M_{\alpha,\gamma} = \sum_{d=1}^{\mathrm{val}(\alpha'(\gamma))} (1-\alpha^\vee)^d \, \mathbb{C}[\tilde{W}] \otimes_{\mathbb{C}} S(\mathfrak{t}) \{\partial_\alpha^d\}$$
$$+ \sum_{d=1}^{\mathrm{val}(\alpha'(\gamma))} (1-\alpha^\vee)^{d-1} (1-w_\alpha) \mathbb{C}[\tilde{W}] \otimes_{\mathbb{C}} S(\mathfrak{t}) \{\partial_\alpha^d\}.$$

$\gamma \in \mathfrak{t}(\mathcal{O})$ regular element.
Assume $H_*(Y_\gamma; \mathbb{C})$ is pure.
Then, $\tilde{W} \subset Y_\gamma$ induces an exact sequence of $\mathcal{D}(\mathfrak{t})$-modules

$$0 \to \sum_{\alpha \in \Phi^+} M_{\alpha,\gamma} \to \mathbb{C}[\tilde{W}] \otimes_{\mathbb{C}} S(\mathfrak{t}) \to H_*^{T(\mathbb{C})}(Y_\gamma) \to 0$$

$(\tilde{W} \times \mathrm{Aut})_\gamma$ acts on this eq. hom grp & restricts to an action on the ordinary homology

$$H_*(Y_\gamma) = H_*^{T(\mathbb{C})}(Y_\gamma)\{I\}$$

Springer action

$$\tilde{W} \curvearrowright^{\text{right}} \mathbb{C}[\tilde{W}] \otimes S(\mathfrak{t})$$

regular $\otimes$ trivial

It preserves each relation $M_{\alpha,\gamma}$ $\alpha \in \Phi^+$.

Assume $H_*(Y_\gamma; \mathbb{C})$ is pure -

induces action of $\mathbb{C}[\tilde{W}]$ on homology compatible w/ the $\mathcal{D}(\mathfrak{t})$ module structure, & commute w/ the $(\tilde{W} \times \mathrm{Aut})_\gamma$ act.

$\leadsto$ It restricts to action of $\tilde{W}$ on ordinary cohomology, coincide w/ Lusztig action.

action of trigonometric DAHA.

Affine flag manifold for $SL(2)$

$G = SL(2)$

$B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$

$I \subset G(\mathcal{O})$ Iwahori

$Y = Y^{SL(2)} = G(F)/I$ affine flag manifold.

$x_0 = I/I \in Y$ base point.

$T \subset G$ torus of diag. matrices

$\Lambda = \{\alpha^\vee(\varepsilon^m) : m \in \mathbb{Z}\}$ translation lattice

$\alpha^\vee \in \Phi^\vee$ simple coroot determined by $T \subset B$.

$T(\mathbb{C}) \curvearrowright Y$ w/ fixed points

$$l_m = \begin{pmatrix} \varepsilon^m & 0 \\ 0 & \varepsilon^{-m} \end{pmatrix} I$$

and $\quad r_m = \begin{pmatrix} 0 & \varepsilon^m \\ -\varepsilon^{-m} & 0 \end{pmatrix} I \quad , \quad m \in \mathbb{Z}$

$W = \{1, w_\alpha\}$ Weyl grp of G.

$\tilde{W} = \Lambda \rtimes W$ acts simply transitively on these fixed points.

$\tilde{W} \simeq$ fixed points

* $\alpha^\vee(\varepsilon^m) \mapsto l_m$
* $w_\alpha \longmapsto r_0 \in Y$

$$\left.\right) \rightsquigarrow \begin{cases} l_a l_b = l_{a+b} \\ l_a r_b = r_{a+b} \\ r_a l_b = r_{a-b} \\ r_a r_b = l_{a-b}. \end{cases}$$

* $x_0 = l_0$ base point
* $y_1 := r_0$.

$\tilde{T}(\mathbb{C})$ extended torus.

If $m \leq -1$, $s \in \mathbb{Z}$, the $\tilde{T}(\mathbb{C})$-orbit of $l_s x_m$ is 1-dim, and connects l_s and r_{m+s}.

$$\pi : \begin{array}{c} Y^{SL(2)} \\ \downarrow \\ X^{SL(2)} \end{array}$$

proj to affine grassmannian.

$G(F)$ equiv fibration w/ fibers $G(\mathbb{C})/B(\mathbb{C}) \simeq \mathbb{P}^1_{\mathbb{C}}$.

$\pi(l_m) = \pi(r_m)$ $\forall m$,

$\pi(x_m) = \pi(y_m)$ $m \leq 0$.

For $m \leq 0$, $T(F) \cdot x_m$ is $(-m)$-dim

projects under π iso to $T(F) \cdot \pi(x_m)$.

For $m \leq 1$, $\dim\left(T(F)\cdot y_m\right) = 1-m$.

If $m \leq 0$, it fibers over the $T(F)$-orbit of $\pi(y_m) = \pi(x_m)$ w/ 1-dim affine spaces fibers

- Fix $\gamma \in \mathfrak{t}(G)$, $Y_\gamma = \{xI \in Y : Ad(x^{-1})(\gamma) \in \mathrm{Lie}(I)\}$

  preserved by $\tilde{T}(\mathbb{C})$,

  $\pi : Y_\gamma \to X_\gamma$ surj. Not a fibration.

  $H_*(Y_\gamma)$ is pure

Prop: $Y = \bigsqcup_{n \leq 0} T(F)\cdot\{x_n \cup y_{n+1}\}$

$$Y_\gamma = Y_{\leq r} = \bigcup_{n=-r}^{0} T(F)\cdot\{x_n \cup y_{n+1}\}$$

$$r = \mathrm{val}(\alpha'(\gamma))$$

$\forall (s,t) \in \mathbb{Z}^2$, $\exists!$ $E_{st} \subset Y$ 1-dim orbit of $\tilde{T}(\mathbb{C})$, connecting l_s and r_{t}, giving all 1-dim orbits of $\tilde{T}(\mathbb{C})$ in Y.

$\tilde{T}(\mathbb{C})$ acts on E_{st} through $(\alpha, s+t)$.

$E_{st} \subset Y_\gamma \iff -r \leq t-s \leq r+1$.

- $\alpha^\vee$ gives $S(\tilde{t}) \cong \mathbb{C}[x,t]$

$$\mathcal{D}(\tilde{t}) \cong \mathbb{C}[\partial_x, \partial_t].$$

$\gamma \in t(G)$, $v = val(\alpha'(\gamma))$.

$M_{ab} \subset \mathbb{C}[\tilde{W}] \otimes_{\mathbb{C}} S(\tilde{t})$ spanned by

$$(l_a - r_b) \otimes g_{ab}((a+b)x - 2t)$$

g_{ab} polynomial.

M_{ab} is $\mathcal{D}(\tilde{t})$-module.

$$Q_v = \sum_{-v \leq b-a \leq v-1} M_{ab}$$

$\tilde{W} \subset Y_\gamma$ induces s.e.s in $T(\mathbb{C})$-equiv coh:

$$0 \to Q_v\{\partial_t\} \to \mathbb{C}[\tilde{W}] \otimes_{\mathbb{C}} S(t) \to H_*^{T(\mathbb{C})}(Y_\gamma) \to 0$$

Proposition: For $v \geq 1$, the module of relations $Q_v\{\partial_t\}$ is spanned by

$$\sum_{d=1}^{v} (1-\alpha^\vee)^d \, \mathbb{C}[\tilde{W}] \otimes S(t)\{\partial_x^d\}$$

and $$\sum_{d=1}^{v} (1-\alpha^\vee)^{d-1}(1-w_\alpha)\, \mathbb{C}[\tilde{W}] \otimes S(\mathfrak{t})\{\partial_\alpha^d\}.$$

Next step: groups of semisimple rank one

H ss rank 1.

$T \subset B \subset H$

$I \subset H(F)$ Iw.

$Y^H = H(F)/I$ aff flag man. for H.

$\alpha, \alpha^\vee \rightsquigarrow w_\alpha \in W$ in Weyl grp of H

$\langle \alpha^\vee \rangle \subset \Lambda^H$

1-dim.

$\partial_\alpha \in \mathcal{D}(\mathfrak{t})$ degree 1.

$\tilde{W} = \Lambda \rtimes W$. extended affine Weyl grp.

$SL(2) \hookrightarrow H$ gives

$$\phi_\alpha : Y^{SL(2)} \longrightarrow Y^H$$

$\gamma \in \mathfrak{t}(\mathcal{O})$ reg.

$v = \mathrm{val}\, \alpha'(\gamma)$.

$$Y^H = \coprod_{\ell \in \Lambda^H / \langle \alpha^\vee \rangle} \ell \cdot \phi_\alpha (Y^{SL(2)})$$

$$Y^H_\gamma = \bigsqcup_{\ell \in \Lambda^H/\langle \alpha^\vee \rangle} \ell\, \phi_\alpha\left(Y^{SL(2)}_{\leq \nu}\right).$$

$$\| $$

$$: Y^H_{\leq \nu} .$$

Prop. We have

$$0 \to \mathbb{Q}_\nu\{\partial_z\} \to \mathbb{C}[\tilde{W}] \otimes_{\mathbb{C}} S(\mathfrak{t}) \to H^{T(\mathcal{O})}_*\left(Y^H_{\leq \nu}\right) \to 0$$