Hyperkähler cohomology and BPS cohomology

EGRET Seminar
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Hyperkähler cohomology

The known Hyperkähler Manifolds

Hodge numbers of Hyperkählers

BPS cohomology

Review of 2CY DT theory (for K3 surfaces)

BPS sheaves and the main conjectures

LLV Lie algebras

Lefschetz modules

LLV Lie algebras and BPS cohomology?

Hyperkähler manifolds

Definition

A (compact) **hyperkähler manifold** $X = (X, \omega)$ is a simply-connected, smooth, projective complex variety together with a non-degenerate, closed holomorphic 2-form $\omega \in H^0(X, \Omega^2_X)$.

- S 2CY surface $\rightsquigarrow$ moduli of coherent sheaves $\mathcal{M}$ of carry a (0-shifted) symplectic form

Example

1. Hilbert schemes of points on a K3 surface $\mathrm{K3}^{[n]} \sim \mathcal{M}_S^{H-\mathrm{st}}(v)$, v primitve
2. generalized Kummer varieties $\mathrm{Kum}^{[n]}$ (arise from abelian surfaces)
3. O'Grady's 10-dimensional example $OG10$
4. O'Grady's 6-dimensional example $OG6$

Hodge numbers of the HKs arising from K3s

Hodge numbers of $K3^{[n]}$ [Göttsche,Göttsche–Soergel,1993]:

$$\sum_{n=1}^{\infty} E(K3^{[n]})t^n = \prod_{m=1}^{\infty} \frac{1}{(1-uvt^m)(1-uv^{-1}t^m)(1-u^{-1}vt^m)(1-(uv)^{-1}t^m)(1-t^m)^{20}}$$

Hodge diamond of $OG10$ [dCRS,GKLR,2019]:

										1										
									0		0									
								1		22		1								
							0		0		0		0							
						1		22		254		22		1						
					0		0		0		0		0		0					
				1		23		276		2299		276		23		1				
			0		0		0		0		0		0		0		0			
		1		22		276		2531		16490		2531		276		22		1		
	0		0		0		0		0		0		0		0		0		0	
1		22		254		2299		16490		88024		16490		2299		254		22		1
	0		0		0		0		0		0		0		0		0		0	
		1		22		276		2531		16490		2531		276		22		1		
			0		0		0		0		0		0		0		0			
				1		23		276		2299		276		23		1				
					0		0		0		0		0		0					
						1		22		254		22		1						
							0		0		0		0							
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										1										

Cohomological DT theory for coherent sheaves on K3 surfaces

S K3 surface over $\mathbb{C}$, H ample divisor on S, $v \in H^\bullet_{\mathrm{alg}}(S,\mathbb{Z})$

$$\begin{array}{ccc} \mathfrak{M}(v) = \mathfrak{M}_S^{H-\mathrm{ss}}(v) & & \text{moduli stack of Gieseker } H\text{-semistable coherent sheaves } F \text{ on } S \text{ with Mukai vector } v \\ \downarrow p & & \\ \mathcal{M}(v) = \mathcal{M}_S^{H-\mathrm{ss}}(v) & & \text{coarse moduli space} \end{array}$$

The pushforward $p_\star \mathbb{D}\underline{\mathbb{Q}}_{\mathfrak{M}}$ is pure $\rightsquigarrow$ less (2D) perverse filtration $\mathcal{L}^\bullet p_\star \mathbb{D}\underline{\mathbb{Q}}_{\mathfrak{M}}$

parallel (conjectural) procedure on $S \times \mathbb{A}^1$ with vanishing cycle cohomology dimensional reduction $\rightsquigarrow$ (3D) perverse filtration $\mathcal{P}^\bullet p_\star \mathbb{D}\underline{\mathbb{Q}}_{\mathfrak{M}}$

BPS sheaves and (Lie) algebras

Definition (conjectural)

Let $v \in H^{\bullet}_{\mathrm{alg}}(S,\mathbb{Z})$. The **BPS sheaf of S in class v** is the perverse sheaf $\mathcal{BPS}(v) = \mathcal{P}^1 p_{\star}\mathbb{D}\underline{\mathbb{Q}}_{\mathfrak{M}(v)}$.

Definition (BPS algebra)

Let $w \in H^{\bullet}_{\mathrm{alg}}(S,\mathbb{Z})$ be a primitive class. The **BPS algebra of slope w** is the perverse sheaf $\mathcal{U}_{\mathrm{BPS}}(w) = \mathcal{L}^0\left(\bigoplus_{r\geq 0} p_{\star}\mathbb{D}\underline{\mathbb{Q}}_{\mathfrak{M}(rw)}\right)$

- $\mathfrak{g}_{\mathrm{BPS}}(w) = \bigoplus_{r\geq 1}\mathcal{BPS}(rw)$ is a Lie algebra
- Relationship between 2D and 3D perverse filtrations:

$$\mathcal{U}_{\mathrm{BPS}}(w) = U\left(\bigoplus_{r\geq 1}\mathcal{BPS}(rw)\right)$$

Expectations for the BPS sheaf

Conjecture (Cohomological Integrality Conjecture)

Let $w \in H^{\bullet}_{\mathrm{alg}}(S, \mathbb{Z})$ be a primitive class. There is an isomorphism

$$\mathcal{H}\mathcal{A}^{H-\mathrm{ss}}(w) := \bigoplus_{r \geq 0} p_{\star} \mathbb{D} \underline{\mathbb{Q}}_{\mathfrak{M}(rw)} \cong \mathrm{Sym}\left(\mathfrak{g}_{\mathrm{BPS}}(w) \otimes H(\mathrm{pt}/\mathbb{C}^{\times})\right)$$

Conjecture (Free Conjecture)

Let $w \in H^{\bullet}_{\mathrm{alg}}(S, \mathbb{Z})$ be a primitive class such that $w^2 \geq 0$. Then

$$\mathcal{U}_{\mathrm{BPS}} = \mathrm{Free}_{\mathrm{Alg}}\left(\bigoplus_{r \geq 0} \mathcal{IC}(\mathcal{M}(rv))\right) = U\left(\mathrm{Free}_{\mathrm{Lie}}\left(\bigoplus_{r \geq 0} \mathcal{IC}(\mathcal{M}(rv))\right)\right)$$

Conjecture (χ-independence conjecture)

See next slide...

The χ-independence conjecture for BPS cohomology

Conjecture (χ-independence for cohomology)

For all curve classes $\beta \in H^2_{\mathrm{alg}}(S,\mathbb{Z})$ and for all $\chi, \chi' \in \mathbb{Z}$ we have

$$\mathrm{BPS}(0,\beta,\chi) \cong \mathrm{BPS}(0,\beta,\chi').$$

For all classes $v, v' \in H^{\bullet}_{\mathrm{alg}}(S,\mathbb{Z})$ with $v^2 = v'^2$ we have $\mathrm{BPS}(v) \cong \mathrm{BPS}(v')$.

Conjecture (χ-independence over the Chow variety)

For all curve classes $\beta \in H^2_{\mathrm{alg}}(S,\mathbb{Z})$ and for all $\chi, \chi' \in \mathbb{Z}$ for the Hilbert–Chow morphisms

$$\mathcal{M}(\beta,\chi) \xrightarrow{\ h\ } \mathrm{Chow}(\beta) \xleftarrow{\ h'\ } \mathcal{M}(\beta,\chi')$$

we have $h_\star \mathcal{BPS}(\beta,\chi) \cong h'_\star \mathcal{BPS}(\beta,\chi')$.

Hodge numbers of $OG10$ from the expectations for BPS sheaves

w primitive such that $w^2 = 2$, $v = 2w$, and v' primitive such that $v^2 = v'^2$.

$$OG10 = \mathrm{Bl}_\Sigma \mathcal{M}(v) \xrightarrow[\text{symp. res.}]{b} \mathcal{M}(v) \xrightarrow{h} \mathrm{Chow}(\beta) \xleftarrow{h'} \mathcal{M}(v')$$

Decomposition theorem for b:

$$b_\star \underline{\mathbb{Q}}_{OG10} \cong \mathcal{IC}(\mathcal{M}(v)) \oplus \mathcal{IC}(\mathrm{Sym}^2(\mathcal{M}(w))) \oplus \mathcal{IC}(\mathcal{M}(w))$$

BPS sheaves:

$$\mathcal{BPS}(v) = \mathcal{IC}(\mathcal{M}(v)) \oplus \Lambda^2 \mathcal{IC}(\mathcal{M}(w)) \qquad \text{(Free conj.)}$$

$$\mathcal{BPS}(v') = \underline{\mathbb{Q}}_{\mathcal{M}(v)} \qquad \text{(no strictly semistables)}$$

$$h_\star \mathcal{BPS}(v) \cong h'_\star \mathcal{BPS}(v') \qquad (\chi\text{-indep.})$$

$\rightsquigarrow$ write $(h \circ b)_\star \underline{\mathbb{Q}}_{OG10}$ in terms of (pushforwards of) constant sheaves on $\mathrm{Sym}^2(\mathcal{M}(w)), \mathcal{M}(w)$ and $\mathcal{M}(v')$, we know $\mathcal{M}(w) \sim \mathrm{K3}^{[4]}, \mathcal{M}(v') \sim \mathrm{K3}^{[5]}$

Hodge numbers of $OG10$ à la de Cataldo–Rapagnatta–Sacca

$w = (0, [C], 1)$ primitive such that $w^2 = 2$, $v = 2w$, and $v' = (0, 2[C], 1)$ primitive such that $v^2 = v'^2$.

$$
\begin{array}{ccccc}
OG10 = \mathrm{Bl}_\Sigma \mathcal{M}(v) & \xrightarrow[\text{symp. res.}]{b} & \mathcal{M}(v) & & \mathcal{M}(v') \\
 & & & \searrow^{h} \quad {}^{h'}\swarrow & \\
 & & & \mathbb{P}^5 &
\end{array}
$$

Decomposition theorem for b:

$$
b_\star \underline{\mathbb{Q}}_{OG10} \cong \mathcal{IC}(\mathcal{M}(v)) \oplus \mathcal{IC}(\mathrm{Sym}^2(\mathcal{M}(w))) \oplus \mathcal{IC}(\mathcal{M}(w))
$$

Strategy [dCRS]:

- find many abelian fibrations associated to the problem
- Ngô support theorem $\rightsquigarrow$ decomposition theorem for h and h' is tractable
- Key difficulty: non-reduced curves
- Compare results for h, h'

$\Longrightarrow$ prove χ-independence in this situation

Hodge numbers of $OG10$ via LLV decomposition of Hyperkähler cohomology à la Green–Kim–Laza–Robles

X smooth, projective $\rightsquigarrow$ LLV Lie algebra $\mathfrak{g}_{\mathrm{LLV}}(X) \subset \mathrm{End}(H^\bullet(X,\mathbb{Q}))$
$\rightsquigarrow$ study $H^\bullet(X,\mathbb{Q})$ as a $\mathfrak{g}_{\mathrm{LLV}}(X)$-module

Theorem ([LLV])

- $\mathfrak{g}_{\mathrm{LLV}}(S) = \mathfrak{so}(4,20)$
- $\mathfrak{g}_{\mathrm{LLV}}(\mathrm{K3}^{[n]}) = \mathfrak{so}(4,21)$ *and* $\mathfrak{g}_{\mathrm{LLV}}(OG10) = \mathfrak{so}(4,22)$
- $\mathfrak{g}_{\mathrm{LLV}}(\mathrm{Kum}^{[n]}) = \mathfrak{so}(4,5)$ *and* $\mathfrak{g}_{\mathrm{LLV}}(OG6) = \mathfrak{so}(4,6)$

Theorem ([GKLR])

Generating series of $\mathfrak{so}(4,21)$-characters:

$$\sum_{n=0}^{\infty} \mathrm{ch}(H^\bullet(\mathrm{K3}^{[n]}))t^n = \prod_{m=1}^{\infty}\prod_{i=1}^{11} \frac{1}{(1-x_i t^m)(1-x_i^{-1}t^m)}$$

As an $\mathfrak{so}(4,22)$-module: $H^\bullet(OG10) = V_{5\varpi_1} + V_{2\varpi_2}$

Question Can one say anything about BPS cohomology using LLV type methods?

Lefschetz modules: definition

k field, $\mathrm{char}(k) = 0$
$M = M^\bullet$ a $\mathbb{Z}$-graded k-vector space, $\dim_k(M) < \infty$
$h : M \to M$ multiplication by d on the degree d part of M

Definition

1. $e \colon M \to M[-2]$ **has the Lefschetz property(LP)** if $\forall d$ $e^d : M^{-d} \to M^d$ is an iso. Equivalently, $\exists f : M \to M[2]$ s.t. $[e, f] = h$ (i.e. (e, h, f) is an $\mathfrak{sl}_2$-triple).
2. $\mathfrak{a} = \mathfrak{a}[-2], \dim_k(\mathfrak{a}) < \infty$, then a graded map $e \colon \mathfrak{a} \to \mathrm{End}(M)$ **has the LP** if $\exists a \in \mathfrak{a}$ s.t. e_a has the LP.
   Define the Lie algebra
   $$\mathfrak{g}(\mathfrak{a}, M) = \langle e_a \mid \forall a \text{ s.t. } e_a \text{ has the LP} \rangle \subset \mathrm{End}(M)$$
3. $(\mathfrak{a}, M)$ is a **Lefschetz module** if $\mathfrak{g}(\mathfrak{a}, M)$ is semisimple.

$M = M_{\mathrm{even}} \oplus M_{\mathrm{odd}}$ as a $\mathfrak{g}(\mathfrak{a}, M)$ module

Lefschetz modules: examples

X smooth, projective of dimension n, L an ample line bundle on X

Hard Lefschetz theorem: $c_1(L)^i \cup\colon H^{n-i}(X,\mathbb{Q}) \xrightarrow{\sim} H^{n+i}(X,\mathbb{Q})$

$\implies$ The map $H^2(X,\mathbb{Q}) \longrightarrow \mathrm{End}(H^\bullet(X,\mathbb{Q})[n]), \alpha \mapsto \alpha\cup$ has the Lefschetz property.

Theorem

The pair $(H^2(X,\mathbb{Q}), H^\bullet(X,\mathbb{Q})[n])$ *is a Lefschetz module.*

Definition

The **LLV Lie algebra** of X is $\mathfrak{g}_{\mathrm{LLV}}(X) = \mathfrak{g}(H^2(X,\mathbb{Q}), H^\bullet(X,\mathbb{Q})[n])$

- for singular X work with $\mathcal{IC}(X)$
- $f\colon X \longrightarrow Y$ proper, L very ample line bundle on X
  Relative hard Lefschetz $\implies$ $c_1(L)^i \cup\colon {}^{\mathfrak{p}}\mathcal{H}^{-i}(f_\star\mathcal{IC}(X)) \xrightarrow{\sim} {}^{\mathfrak{p}}\mathcal{H}^{i}(f_\star\mathcal{IC}(X))$
  notion of Lefschetz constructibe complex/graded perverse sheaf?

Lefschetz modules: applications to BPS cohomology?

Disclaimer: the following points are basically daydreams

- BPS is a finite dimensional graded vector space, can we view it *a priori* as a Lefschetz module? Maybe suggested by $\mathfrak{sl}_2 \times \mathfrak{sl}_2$-action that appears in the theory of Gopakumar–Vafa invariants?

  A posteriori from the Free Conjecture: Yes, because $\mathcal{BPS}$ is built out of $\mathcal{IC}$-sheaves.

- Is it worth thinking about LLV-type ideas over the Chow-variety instead of in cohomology/over a point?

- Can we do LLV-type stuff in the world of Higgs bundles?
- Taelman used an (interpretation of) the LLV Lie algebra coming from the Hochschild homology to study derived equivalences between hyperkählers. Connections?